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ON THE CONSTRUCTION OF FINITE DIFFERENCE SCHEMES APPROXIMATING GENERALIZED SOLUTIONS

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Abstract. We consider Dirichlet's problem for Poisson's equation in n-dimensional Euclidean space assuming that the generalized solution belongs to the Sobolev space $W^{s,p}$, $1 \leq s \leq 4$, $1 < p < \infty$. We construct finite difference schemes converging to the generalized solution in integer order discrete Sobolev-like norms.

1. Introduction. Recently there have been many theoretical advances in constructing finite difference schemes approximating generalized solutions of boundary value problems. For example, Lazarov [**4**] presents a finite difference approximation of Dirichlet's problem for Poisson's equation with a generalized solution belonging to the integer order Sobolev space $W^{s,2}$, $s = 2, 3$ and proves that it is convergent in discrete norms using the so called Bramble-Hilbert Lemma [**1**].

Unfortunately, the Bramble-Hilbert Lemma is stated only for integer order Sobolev spaces. Recently Dupont and Scott [**3**] gave a constructive proof of this Lemma using averaged Taylor series and extended it to fractional order Sobolev spaces.

In this paper a basic framework is given which allows the application of the finite difference method in order to approximate generalized solutions belonging to the Sobolev space $W^{s,p}$, $1 \leq s \leq 4$, $1 < p < \infty$.

For simplicity, the analysis in this paper deals only with Dirichlet's problem for Poisson's equation in rectangular domains. Extensions to other elliptic boundary value problems in less special domains are possible.

2. Preliminaires and Notations. Let $\mathcal{A}$ be an open set in n-dimensional Euclidean space R^n with the restriced cone property and $1 < p < \infty$. Throughout the paper $W^{s,p}(\mathcal{A})$ is the Sobolev space of order $s \geq 0$ [**8**] equipped with the Sobolev norm

$$\|u\|_{s,p\mathcal{A}} = \left(\sum_{k=0}^{s} |u|_{k,p,\mathcal{A}}^p\right)^{1/p} \quad \text{with} \quad |u|_{k,p,\mathcal{A}} = \left(\sum_{|\alpha|=k} \|D^\alpha u\|_{L^p(\mathcal{A})}^p\right)^{1/p},$$

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if s is integer and

$$\|u\|_{s,p\mathcal{A}} = (\|u\|^p_{[s],p,\mathcal{A}} + |u|^p_{s,p,\mathcal{A}})^{1/p}$$

if $s = [s] + \sigma$, with $[s]$ =integral part of s, $0 < \sigma < 1$ and

$$|u|_{s,p\mathcal{A}} = \left(\sum_{|\alpha|=[s]} \int_{\mathcal{A}} \int_{\mathcal{A}} \frac{|D^\alpha u(x) - D^\alpha n(y)|^p}{|x-y|^{n+\sigma p}} \delta x \delta y \right)^{1/p}.$$

$\mathbf{N}$ will stand for the set of nonnegative integers. $\mathbf{P}^l(\mathcal{A})$ will denote the set of polynomials in n variables of degree $\leq l$ over the set $\mathcal{A}$, for any $l \in \mathbf{N}$.

The following lemma is an easy consequence of Theorem 6.1 of [**3**] (the case $\sigma = 1$, $p = 2$ follows from the Bramble-Hilbert Lemma [**1**]).

LEMMA. *Suppose that $s = l + \sigma$, where $0 < \sigma \leq 1$ and $l \in \mathbf{N}$. Let η be a bounded linear functional on $W^{s,p}(\mathcal{A})$ such that $\mathbf{P}^l(\mathcal{A}) \subset$ kernel (η). There exists a positive constant $\mathbf{C}$ (depending on $\mathcal{A}, s$ and p) such that for any $u \in W^{s,p}(\mathcal{A})$ $|\eta(u)| \leq \mathbf{C}|u|_{s,p\mathcal{A}}$.*

Remark 1. This lemma may also be proved using either Tartar's (unpublished) Lemma [**2**] or Peetre's Lemma [**5**].

Let $\mathcal{D}'(\Omega)$ denote the space of distributions on Ω for any open set $\Omega \subset \mathbf{R}^n$ and Δ the Laplace operator on $\mathcal{D}'(\Omega)$. We shall assume for the sake of simplicity that Ω is an open rectangle in $\mathbf{R}^n$ and consider the following boundary value problem:

Given $f \in W^{-1,p}(\Omega)$, find a function u that satisfies

(1) $\Delta u = -f$ in Ω, in the sense of distributions

(2) $u = 0$ on $\partial\Omega$, in the sense of trace theorems.

By changing variables, we may assume, without loss of generality, that the rectangle is $\Omega = (0,1)^n$.

3. Mollifiers. Let us consider the function

$$S_\nu(x) = \begin{cases} \left(\dfrac{\sin(x/2)}{x/2}\right)^\nu, & x \neq 0 \\ 1, & x = 0, \end{cases}$$

with $\nu \in \mathbf{N}$. By the Paley-Wiener-Schwartz Theorem [**7**] there exists a distribution Θ_ν with compact support and with a Fourier transform equal to S_ν.

Remark 2. An easy argument shows that Θ_0 is the Dirac distribution, Θ_1 the characteristic function of the interval $(-1/2, 1/2)$ and

$$\Theta_2(x) = \begin{cases} 1 - |x|, & x \in (-1,1) \\ 0, & x \notin (-1,1) \end{cases}$$

Let $\nu = (\nu_1, \ldots, \nu_n) \in \mathbf{N}^n$, $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, G_ν a distribution defined by $G_\nu(x) = h^{-n}\Theta(x/h)$ with Θ_ν the tensor product of distributions

$\Theta_{\nu i}$, $i = 1, \dots n$ and h a positive parameter. For $u \in D'(R^n)$ the operator T_ν given by $T_\nu u = u * G_\nu$ will be called mollifier.

Let $u \in \mathcal{D}'(\Omega)$ and $u^* \in \mathcal{D}'(\mathbf{R}^n)$ any extension of u. $T_\nu u$ will denote the restriction of $T_\nu u^*$ to the rectangle $\Omega_\nu = \{x \in \mathbf{R}^n; h\nu_i/2 < x_i < 1 - h\nu_i/2,\ i = 1, \dots, n\}$. Finally let us observe that $t_\nu u$ does not depend on u^* thus it is well defined.

4. Construction of Finite Difference Schemes. Let $n_0 \geq 2$ be an integer and $h = 1/n_0$. We define the following grids:

$$
\begin{aligned}
R_h^n =& \{x = (x_1^{(i_1)}, \dots, x_n^{(i_n)}) \in \mathbf{R}^n : x_j^{(i_j)} = i_j \cdot h,\ |i_j| < \infty,\ \ j = 1, \dots, n\} \\
\omega_h =& \Omega \cap R_h^n,\ \gamma_h = \partial\Omega \cap R_h^n,\ \overline{\omega}_h = \omega_h \cup \gamma_h, \\
\gamma_h^j =& \gamma_h \cap ((0,1)^{j-1} \times \{0,1\} \times (0,1)^{n-j}),\ j = 1, \dots, n \\
\gamma_h^{n+1} =& \gamma_h \cap \cup_{j=1}^n (0,1)^{j-1} \times \{0\} \times (0,1)^{n-j} \\
\omega_h^+ =& \omega_h \cup \gamma_h^{n+1}.
\end{aligned}
$$

For ν function of discrete arguments defined on R_h^n, set

$$
(\Delta_j \nu)(x) = \frac{\nu(x + e_j h) - \nu(x)}{h},\ (\nabla_j \nu)(x) = \frac{\nu(x) - \nu(x - e_j h)}{h},
$$

with $e_j = (\delta_{1j}, \dots, \delta_{nj})$ and define the discrete Laplace operator Δ_h by

$$
\Delta_h \nu = \sum_{j=1}^n \Delta_j \nabla_j \nu.
$$

Finally let us introduce the following discrete norms:

$$
\begin{aligned}
\|\nu\|_{0,p,h} =& \Big(h^n \sum_{x \in \omega_h} |\nu(x)|^p \Big)^{1/p} \\
|[\nu\|_{0,p,h} =& \Big(h^n \sum_{x \in \omega_h^+} |\nu(x)|^p \Big)^{1/p} \\
\|\nu\|_{1,p,h} =& \Big(\|\nu\|_{0,p,h}^p + \sum_{j=1}^n |[\Delta_j \nu\|_{0,p,h}^p \Big)^{1/p} \\
\|\nu\|_{2,p,h} =& \Big(\|\nu\|_{1,p,h}^p + \sum_{i \neq j} |[\Delta_i \Delta_j \nu\|_{0,p,h}^p + \sum_{i=1}^n \|\Delta_i \nabla_i \nu\|_{0,p,h}^p \Big)^{1/p}.
\end{aligned}
$$

Consider the following finite difference scheme

$$
-\Delta_h z = \sum_{j=1}^n \Delta_j \nabla_j \eta_j, \qquad x \in \omega_h \tag{3}
$$

$$
z(x) = 0, \qquad x \in \gamma_h \tag{4}
$$

with η_j defined on $\omega_h \cup \gamma_h^j$ and equal to zero on γ_h^j, $j = 1, \dots, n$. An easy argument based on the discrete multiplicator techniques shows that

$$\|z\|_{0,p,h} \le C \sum_{j=1}^{n} \|\eta_j\|_{0,p,h} \tag{5}$$

$$\|z\|_{1,p,h} \le C \sum_{j=1}^{n} |[\Delta_j \eta_j\|_{0,p,h} \tag{6}$$

$$\|z\|_{2,p,h} \le C \sum_{j=1}^{n} \|\Delta_j \nabla_j \eta_j\|_{0,p,h} \tag{7}$$

with C positive constant independent of z and h.

Boundary value problem (1), (2) has a unique solution $u \in W_0^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) : u = 0 \text{ on } \partial\Omega\}$, $1 < p < \infty$. Let $\Omega^* = (-1, 2)^n$. Extension of u by 0 outside Ω is a continuous mapping of $W_0^{1,p}(\Omega)$ into $W^{1,p}(\Omega^*)$ [**6**]. Hence,

$$u \to u^* = \text{odd extension of } u$$

is a continuous mapping of $W_0^{1,p}(\Omega)$ into $W^{1,p}(\Omega^*)$. Let $f^* = \Delta u^*$ and $e = (1, \dots, 1)$. Applying T_{2e} we have

$$\sum_{j=1}^{n} \Delta_j \nabla_j T_{2e-2e_j} u^* = -(T_{2e} f^*)(x), \qquad x \in \omega_h.$$

$T_{2e} f^*$ is a continuous function on Ω_{2e} and $T_{2e} f^* = T_{2e} f$ on Ω_{2e}. Thus,

$$\sum_{j=1}^{n} \Delta_j \nabla_j T_{2e-2e_j} u^* = -(T_{2e} f)(x), \quad x \in \omega_h \tag{8}$$

Similarly, $T_e u^*$ is a continuous function on Ω_e, $T_e u^* = T_e u$ on Ω_e and

$$(T_e u^*)(x) = 0, \quad x \in \gamma_h. \tag{9}$$

We associate to (1), (2) the finite difference scheme

$$\Delta_h \nu = -(T_{2e} f)(x), \quad x \in \omega_h \tag{10}$$

$$\nu(x) = 0, \quad x \in \gamma_h. \tag{11}$$

5. Convergence of the Finite Difference Scheme

Theorem. *Let u be the solution of boundary value problem* (1), (2), *ν the solution of discrete problem* (10), (11) *and $k \in \{0, 1, 2\}$. If u belongs to $W^{s,p}(\Omega)$ with $1 \le s \le k + 2$ and $1 < p < \infty$, the following error estimate holds*

$$\|T_e u - \nu\|_{k,p,h} \le C h^{s-k} |u|_{s,p,\Omega},$$

with a positive constant C independent of h. Moreover, if $s \geq k$ then finite difference scheme (10), (11) *converges in the dicrete norm* $\|\cdot\|_{k,p,h}$.

Proof. We shall give the proof for $k=0$. The procedure is similar for $k=1$ and $k=2$. By (8)—(11) function $z = \nu - T_e u^*$ is defined on $\overline{\omega}_h$ and satisfies (3), (4) with $\eta_j = T_e u^* - T_{2e-2e_j} u^*$. Function η_j is defined on $\omega_h \cup \gamma_h^j$ and equal to zero on γ_h^j. Thanks to inequality (5) it suffices to estimate $\|\eta_j\|_{0,p,h}$, $j=1,\dots,n$. For $1 \leq i_j \leq n_0 - 1$ we introduce the squares

$$E(i_1,\dots,i_n) = \{x = (x_1,\dots,x_n) \in \mathbf{R}^n : (i_j - 1)h < x_j < (i_j+1)h,\ \ j=1,\dots,n\}$$
$$E = \{t = (t_1,\dots,t_n) \in \mathbf{R}^n : -1 < t_j < 1,\ \ j = 1,\dots,n\},$$

and the affine mapping
$x = (x_1,\dots,x_n) \in E(i_1,\dots,i_n) \to t = (t_1,\dots,t_n) \in E$, with $x_j = i_j h + t_j h$, $j = 1,\dots,n$. Set $\tilde{u}(t) = u^*(x(t))$. Then,

$$\eta_j(i_1 h,\dots,i_n h) = \int_{-1/2}^{1/2}\cdots\int_{-1/2}^{1/2} \tilde{u}(t_1,\dots,t_n)\delta t_1\cdots\delta t_n -$$
$$-\int_{-1}^{1}\cdots\int_{-1}^{1}\Theta_{2e-2e_j}(t_1,\dots,t_n)\tilde{u}(t_1,\dots,t_n)\delta t_1\cdots\delta t_{j-1}\delta t_{j+1}\cdots\delta t_n.$$

Furthermore, $\eta_j(i_1 h,\dots,i_n h)$ is a bounded linear functional on $W^{s,p}(E)$, $s \geq 1$ and $P^1(E) \subset$kernel $(\eta_j(i_1 h,\dots,i_n h))$. By the lemma,

$$|\eta_j(i_1 h,\dots,i_n h) \leq |C| \tilde{u}|_{s,p,E}, \quad 1 \leq s \leq 2,$$

thus

$$|\eta_j(i_1 h,\dots,i_n h| \leq C h^{s-n/p} |u^*|_{s,p,E(i_1,\dots,i_n)}, \quad 1 \leq s \leq 2.$$

Finally,

$$\|\eta_j\|_{0,p,h} \leq C h^s |u|_{s,p,\Omega},\ \ 1 \leq s \leq 2$$

and

$$\|T_e u - \nu\|_{0,p,h} = \|T_e u^* - \nu\|_{0,p,h} = \|z\|_{0,p,h} \leq C h^s |u|_{s,p,\Omega}, \quad 1 \leq s \leq 2.$$

That completes the proof.

Remark 3. Introducing fractional order discrete Sobolev like norms $\|\cdot\|_{k,p,h}$ and using the discrete interpolation technique it is possible to show that the statement of the theorem holds for all k, $0 \leq k \leq 2$.

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