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# THE GENERAL LINEAR EQUATION ON VECTOR SPACES

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**Abstract**. General solution of linear equation of the form (1) and (3) are obtained by means of the generalized inverse functions. The obtained theorems are applied to equations on near-rings, linear functionals, matrix, differential and functional equations.

## 1. General theorems

Let $X$ and $Y$ be nonempty sets and let $f : X \to Y$ be a surjection. The existence of a function $g : Y \to X$ such that $(\forall y \in Y) f(g(y)) = y$ is a well-known equivalent of the Axiom of Choice, due to Bernays [**1**] (see also [**2**]). By a slight modification of the argument, we prove the following

THEOREM 1. *Suppose that $X$ and $Y$ are nonempty sets, and let $a \in X$, $b \in Y$. If $f : X \to Y$ and $f(a) = b$, then there exists a function $g : Y \to X$ such that:*

(i) *$fgf = f$, i.e. $(\forall x \in X) f(g(f(x))) = f(x)$;*

(ii) *$g(b) = a$.*

*Proof*. If $f(X)$ is a singleton, then $f(X) = \{b\}$ and the function $g : Y \to X$ defined by $(\forall y \in Y) g(y) = a$ satisfies (i) and (ii).

If $f(X)$ contains more than one element, let $X_y = \{x \mid f(x) = y\}$, where $y \in f(X)$. Then $X_y \neq \varnothing$, $f(X)\backslash\{b\} \neq \varnothing$, and according to the Axiom of Choice there exists a function $G : f(X)\backslash\{b\} \to X\backslash X_b$ such that $G(y) \in X_y$. The function $g : Y \to X$ defined by

$$g(y) = \begin{cases} G(y), & y \in f(X)\backslash\{b\} \\ a, & y = b \\ H(y), & y \in Y\backslash f(X) \end{cases}$$

where $H : Y\backslash f(X) \to X$ is an arbitrary function (for example, $H(y) = a$, for every $y \in Y\backslash f(X)$) satisfies the conditions (i) and (ii).

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Indeed, (ii) is trivial. To prove (i) notice that for arbitrary $x \in X$ we have

$$g(f(x)) = \begin{cases} G(f(x)), & x \in X \backslash X_b (\Leftrightarrow f(x) \neq b) \\ a, & x \in X_b \quad (\Leftrightarrow f(x) = b) \end{cases}$$

and so

$$f(g(f(x))) = \begin{cases} f(x), & f(x) \neq b \\ a, & f(x) = b \end{cases} = f(x)$$

which completes the proof.

We now apply Theorem 1 to the general linear equation on groups. Namely, suppose that $(G_1, *)$ and $(G_2, \circ)$ are groups whose neutral elements are denoted by $e_1$ and $e_2$, respectively. If $f : G_1 \to G_2$ is a homomorphism, then $f(e_1) = e_2$, and hence, according to Theorem 1 there exists a function $g : G_2 \to G_1$ such that $fgf = f$ and $g(e_2) = e_1$.

Theorem 2. *Consider the equation in x:*

$$f(x) = e_2 \tag{1}$$

*The general solution of the equation* (1) *is given by:*

$$x = t * \overline{g(f(t))}, \tag{2}$$

*where $t \in G_1$ is arbitrary and $\overline{u}$ denotes the inverse of $u (\in G_1$ or $G_2)$.*

*Proof*. The proof of this statement is straight forward. Namely, since $f$ is a homomorphism, from (2) follows

$$f(x) = f(t * \overline{g(f(t))} = f(t) \circ f\overline{(g(f(t)))} = f(t) \circ \overline{f(g(f(t)))} = f(t) \circ \overline{f(t)} = e_2,$$

which means that (2) is a solution of (1). Conversely, suppose that $x_0$ is a solution of (1), i.e. that $f(x_0) = e_a$. Then, putting $t = x_0$ into (2) we get

$$x = x_0 * \overline{g(f(x_0))} = x_0 * \overline{g(e_2)} = x_0 * \overline{e_1} = x_0 * e_1 = x_0.$$

In other words, the solution $x_0$ of (1) is obtained from (2) by putting $t = x_0$, which means that (2) is the general solution of (1).

Consider now the nonhomogeneous equation in $x$:

$$f(x) = c \tag{3}$$

where $c \in G_2$ is given. The equation (3) has a solution if and only if

$$f(g(c)) = c \tag{4}$$

In that case the general solution of (3) is given by

$$x = t * \overline{g(f(t))} * g(c) \tag{5}$$

where $t \in G_1$ is arbitrary. Indeed, if (3) has a solution, then from (3) follows $g(f(x)) = g(c)$, and again $f(g(f(x))) = f(g(c))$. But $fgf = f$ which together with (3) and the last equality implies (4). Conversely, if (4) holds, then $g(c)$ is clearly a solution of (3). The fact that (5) is the general solution of (3) is easily verified.

*Remark*. If $g$ is the inverse function of $f$ then (1) and (3) have unique solutions, namely: $e_1$ and $g(c)$, respectively.

*Problem*. According to Theorem 1, for a homomorphism $f : G_1 \to G_2$ there exists a function $g : G_2 \to G_1$ such that $fgf = f$ and $g(e_2) = e_1$. What additional conditions, if any, are needed to ensure that $g$ is also a homomorphism?

## 2. The case of vector spaces

If $V_1$ and $V_2$ are vector spaces over a scalar fields $S$ and if $f \in \mathrm{Hom}(V_1, V_2)$, i. e. $f : V_1 \to V_2$ is a homomorphism, then there exists a function $g : V_2 \to V_1$ such that $fgf = f$ and $g(0) = 0$, and we obtain corresponding conclusions about the equations $f(x) = 0$ and $f(x) = c$.

However, in this case it is possible to obtain the form of the general solution of those equations. Namely, we have

Theorem 3. *If $f \in \mathrm{Hom}(V_1, V_2)$, the general solution of the equation $f(x) = 0$ has the form $x = h(t)$, where $h \in \mathrm{Hom}(V_1, V_1)$ and $t \in V_1$ is arbitrary.*

*Proof*. We first prove that there exists a homomorphism $g : f(V_1) \to V_1$ such that $fgf = f$. Indeed, since $f(V_1)$ is a vector space, it has a basis $B = \{b_1, b_2, \dots\}$. Moreover, $b_i \in f(V_1)$ and so the set $X_i = \{x \mid f(x) = b_i\}$ is not empty. Hence, according to the Axiom of Choice, there exists a function $g : B \to V_1$ such that $g(b_i) = g_i \in X_i$. For arbitrary $y = \sum_{k=1}^{n(y)} \alpha_k b_k \in f(V_1)$ define $g(y) = \sum_{k=1}^{n(y)} \alpha_k g_k$. The function $g : f(V_1) \to V_1$ defined in this way is clearly a homomorphism and it is easily verified that for all $x \in V_1$ we have $f(g(f(x))) = f(x)$. Hence, the general solution of $f(x) = 0$ is $x = t - g(f(t)) = (i - gf)(t)$, where $i : V_1 \to V_1$ is the identity mapping and $t \in V_1$ is arbitrary. Since $h = i - gf \in \mathrm{Hom}(V_1, V_1)$, the theorem is proved.

Therefore, the general solution of the linear equation $f(x) = 0$ is a linear function of an arbitrary element $t$. However, in order to obtain the solution explicitly. it is necessary to construct the function $g$.

## 3. Applications

We now investigate some cases in which the function $g$ can be determined.

**3.1. Linear equations on near-rings**. Suppose that $(P, +, \cdot)$ is a near-ring (i. e. the group $(P, +)$ need not be commutative). The function $f : P \to P$ defined by $f(x) = axb$, where $a, b \in P$ are fixed, is a homomorphism.

If $a, b$ are regular elements of $P$, i. e. if there exist $\overline{a}, \overline{b} \in P$, such that $a\overline{a}a = a$, $b\overline{b}b = b$, then the function $g : P \to P$ defined by $g(x) = \overline{a}x\overline{b}$ is such that $fgf = f$. Hence, the general solution of the equation $axb = 0$ is: $x = t - \overline{a}atb\overline{b}$. The nonhomogeneous equation $axb = c$ has a solution if and only if $a\overline{a}c\overline{b}b = c$; in that case, the general solution is $x = t - \overline{a}atb\overline{b} + \overline{a}c\overline{b}$. For instance, the general solution of $axb = ab$ is: $x = t - \overline{a}atb\overline{b} + \overline{a}ab\overline{b}$, where $t \in P$ is arbitrary.

More general equations, together with applications to matrix equations are considered in [**3**].

**3.2. Linear functionals**. Let $V$ be a vector space overr the field $S$, let $f : V \to S$ be a linear functional on $V$ and consider the equation in $x$:

$$f(x) = 0 \tag{6}$$

We suppose that there exists $x_0 \in V$ such that $f(x_0) \neq 0$; otherwise (6) holds for all $x \in V$.

For the function $g : S \to V$ defined by $g(s) = sx_0/f(x_0)$ it is easily verified that $fgf = f$. Hence, the general solution of (6) is $x = t - x_0 f(t)/f(x_0)$, where $t \in V$ is arbitrary. Moreover, the general solution of the nonhomogeneous equation $f(x) = c$ is: $x = t + (c - f(t))x_0/f(x_0)$, where $t \in V$ is arbitrary.

Various applications of this result, particularly to integral equations, are given in [**4**].

**3.3. The function $f$ satisfies a polynomial equation**. Let $f : V \to V$, where $V$ ia a vector space over a field $S$ and suppose that the function $f$ satisfies an equation of the form:

$$\lambda_n f^n + \lambda_{n-1} f^{n-1} + \cdots + \lambda_1 f + \lambda_0 i = 0, \tag{7}$$

where $\lambda_0, \ldots, \lambda_n \in S$, $i : V \to V$ is the identity mapping and $f^k$ is the $k$-th iterate of $f$. We have the following conclusions:

(i) If $\lambda_0 \neq 0$, then the function $g$ defined by

$$g = -\lambda_0^{-1}(\lambda_n f^{n-1} + \lambda_{n-1} f^{n-2} + \cdots + \lambda_1 i)$$

is the inverse of $f$.

(ii) If $\lambda_0 = 0$, $\lambda_1 \neq 0$, then the function $g$ defined by

$$g = -\lambda_1^{-1}(\lambda_n f^{n-2} + \lambda_{n-1} f^{n-3} + \cdots + \lambda_2 i)$$

is such that $fgf = f$.

Hence, in those cases it is possible to write down the general solutions of the equations $f(x) = 0$ and $f(x) = c$.

*Remark*. If $\lambda_0 = \lambda_1 = 0$, then $x = \lambda_n f^{n-1}(t) + \cdots + \lambda_2 f(t)$, where $t \in V$ is arbitrary, is clearly a solution of the equation $f(x) = 0$, but examples can be constructed to show that this solution need not be general.

In particular, if $f$ can be written in the form

$$f(x)=\sum_{\nu=1}^{m}\sigma_{1\nu}A_\nu(x)\qquad(\sigma_{1\nu}\in S),\tag{8}$$

where the linear functions $A_1,\dots,A_m:V\to V$ form a semigroup, then

$$f^k(x)=\sum_{\nu=1}^{m}\sigma_{k\nu}A_\nu(x)\qquad(k=1,\dots,m),\tag{9}$$

and eliminating the $A_\nu(x)$'s between (8), (9) and $i(x)=x$, we arrive at an equation of the form (7).

This method was applied in [**5**] to the linear matrix equation

$$A_1XB_1+\cdots+A_mXB_m=0.$$

**3.4. Differential equations**. This example shows how the existing theory of linear differential equations can be interpreted within the framework of the general method given here. Namely, it can be shown [**6**] that the differential equation

$$y''+p(x)y'+q(x)y=0$$

is equavilent to the equation

$$y-\frac{W(y,\psi)}{W(\varphi,\psi)}\varphi-\frac{W(\varphi,y)}{W(\varphi,\psi)}\psi=0,$$

where $\varphi$ and $\psi$ are linearly independent solutions of (10) and $W(u,\nu)=u'\nu-u\nu'$. However, for the function $f$ defined by

$$f(y)=y-\frac{W(y,\psi)}{W(\varphi,\psi)}\varphi-\frac{W(\varphi,y)}{W(\varphi,\psi)}\psi$$

we have $f^2=f$, and hence the general solution of $f(y)=0$, i. e. the general solution of (10) is

$$y=t-f(t)\quad(t\ \text{arbitrary twice differentiable function})\ \text{i.e.}$$

$$y=\frac{W(t,\psi)}{W(\varphi,\psi)}\varphi+\frac{W(\varphi,t)}{W(\varphi,\psi)}\psi.$$

Since it can be shown that the expressions $W(t,\psi)/W(\varphi,\psi)$ and $W(\varphi,t)/W(\varphi,\psi)$ do not depend on $x$ (provided that $p$ has a primitive function), the last expression takes the familiar form: $y=C_1\varphi+C_2\psi$, where $C_1$ and $C_2$ are arbitrary constants.

This method of approach to linear differential equations has certain advantages over the standard method. They are discussed in [**6**].

**3.5. Equations on algebras**. Suppose that $V$ is a commutative algebra, and cosider the equation in $x \in V$:

$$a_{11} A_1 x + \cdots + a_{1n} A_n x = 0, \tag{11}$$

where $a_{11}, \dots, a_{1n} \in V$, $A_1, \dots, A_n : V \to V$ are linear functions with the properties:

(i) $G = \{A_1, \dots, A_n\}$ is a group of order $n$;

(ii) $A_i(\nu A_j x) = A_i(\nu) A_i(A_j x)$ for all $x, \nu \in V$ and $i, j = 1, \dots, n$, Then, if we put

$$f(x) = \sum_{\nu=1}^{n} a_{1\nu} A_\nu x, \tag{12}$$

it again follows that

$$f^k(x) = \sum_{\nu=1}^{n} a_{k\nu} A_\nu x \qquad (k = 1, \dots, n), \tag{13}$$

and again eliminating the $A_\nu x$'s between (12), (13) and $i(x) = x (i \in G)$, we arrive at an equation of the form

$$a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0 i = 0.$$

Though the coefficients $a_0, \dots, a_n$ belong to $V$, it can be shown, by a technique similar to Prešić's [**7**] that $f(a_k f^k) = a_k f^{k+1}$, and so the function $g$ can be formed analogously as in 3.3. The fact that $f(a_k f^k) = a_k f^{k+1}$ corresponds to the condition "compatible with the group $G$" which appears in [**7**].

*Remark*. The equation (11) can be treated in the same way as Prečić [**7**] solved its special case, the equation for $\varphi : E \to K$

$$a_1(x)\varphi(g_1 x) + \cdots + a_n(x)\varphi(g_n x) = 0,$$

where $g_1, \dots, g_n : E \to E$ form a group of order $n$. In this case $E$ is a nonempty set, $K$ is field and $a_1, \dots, a_n : E \to K$.

*Example*. As an example, we solve the following functional equation

$$a(x)\varphi(x) + b(x)\varphi(-x) = 0, \tag{14}$$

where $a, b : R \to R$ are given, and $\varphi : R \to R$ is the unknown function. Let $f : R^R \to R^R$ be defined by

$$f(\varphi(x)) = a(x)\varphi(x) + b(x)\varphi(-x). \tag{15}$$

Then

$$\begin{aligned} f^2(\varphi(x)) = &(a(x)^2 + b(x)b(-x))\varphi(x)) \\ &+ (a(x)b(x) + a(-x)b(x))\varphi(-x), \end{aligned} \tag{16}$$

and elimination of $\varphi(x)$ and $\varphi(-x)$ between (15), (16) and $i(\varphi x) = \varphi(x)$ leads to the equation

$$f^2 - (a(x) + a(-x))f + (a(x)a(-x) - b(x)b(-x))i = 0. \tag{17}$$

If $a(x)a(-x) \neq b(x)b(-x), f$ has its inverse $f^{-1}$ and $\varphi(x) \equiv 0$ is the only solution of (14). Suppose that $a(x)a(-x) = b(x)b(-x)$ and that $a(x) + a(-x) \neq 0$. Then (17) reduces to

$$f^2 - (a(x) + a(-x))f = 0,$$

and the function $g : R^R \to R^R$ defined by

$$g = (a(x) + a(-x))^{-1} i$$

is such that $fgf = f$, which is easily verified. Hence, the general solution of (14) is

$$\varphi(x) = t(x) - \frac{a(x)t(x) + b(x)t(-x)}{a(x) + a(-x)}$$

i. e.

$$\varphi(x) = \frac{a(-x)t(x) - b(x)t(-x)}{a(x) + a(-x)} \quad (t : R \to R \text{ is arbitrary}).$$

* * *

The research which lead to [**3**]—[**6**] and finally to this paper, was initiated mainly by [**7**] and [**8**].

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