PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 37 (51), 1985, pp. 93–99

BASES FROM ORTHOGONAL SUBSPACES OBTAINED BY EVALUATION OF THE REPRODUCING KERNEL

Dušan Georgijević

Abstract. Every inner operator function θ with values in $B(E, E)$, E – a fixed (separable) Hilbert space, determines a co-invariant subspace $H(\theta)$ of the operator of multiplication by z in the Hardy space H_E^2. "Evaluating" the reproducing kernel of $H(\theta)$ at "U-points" of the function θ (U is unitary operator) we obtain operator functions $\gamma_t(2)$ and subspaces $\gamma_t E$. The main result of the paper is: Let the operator $I - \theta(z)U^*$ have a bounded inverse for every z, $|z| < 1$. If $(1-r)^{-1}\Re\varphi(rt)$ for definition of φ see (1) is uniform bounded in r, $0 \leq r < 1$, for all t, $|t| = 1$, except for a countable set, then the familly of subspaces $\gamma_t E$ is orthogonal and complete in $H(\theta)$. This generalizes an analogous result of Clark [**3**] in the scalar case.

1. Introduction. Throughout this paper we denote by D the unit disc $| z |< 1$ and by T the unit circe $| z |= 1$ of the complex plane C. Given a separable Hilbert space $E(E \neq \{0\})$, let H_E^2 be the standard Hardy space of analytic E-valued functions on D. (See [**1**] or [**2**] for general references.) Writting inner products and norms in H_E^2 we will omit designation of the space in the index. The space H_E^2 possesses a so-called reproducing kernel. This is the function $k_w(z) = (1 - z\bar{w})^{-1}$, $w \in D$, $z \in D$, with the following properties: $k_w a \in H_E^2$, $w \in D$, $a \in E$, $(k_w a = k_w(\cdot)\ a)$ and $(f, k_w a) = (f(w),\ a)_E$, $f \in H_E^2$, $w \in D$. If θ is an inner operator function [**1**] (defined on D and with values in $B(E, E)$), then let $H = H(\theta) = H_E^2 \ominus \theta H_E^2$. The reproducing kernel for the space H is the function $K_w(z) = (1 - z\bar{w})^{-1}(I - \theta(z)\theta(w)^*)$, $w \in D$, $z \in D$, where by I is denoted the identity mapping in E.

If U is a unitary operator in E, then we will also consider the following operator functions:

$$\varphi(z) = \varphi_U(z) = (I + \theta(z)U^*)(I - \theta(z)U^*)^{-1}, \tag{1}$$

$z \in D$, (if $(I - \theta(z)U^*)^{-1}$ exists) and

$$\gamma_t(z) = \gamma_U(t, z) = (1 - z\bar{t})^{-1}(I - \theta(z)U^*), \tag{2}$$

AMS Subject Classification (1980): Primary 46 E 40.

$t \in T,\ z \in D$. In the scalar case ($\dim E = 1$) U is a number of modulus 1 and U^* shall be replaced by $\overline{U}$.

In [**3**] Clark considered orhogonal sets in H obtained by evaluation of the kernel $K_w(z)$ on T, in the case $\dim E = 1$. The purpose of this paper is to generalize the criterion for completeness of such orhogonal sets which is contained in Theorem 7.1 of [**3**].

2. Bases from subspaces. Let T_U be the set of all points $t \in T$ such that $\gamma_t a \in H$ for some $a \in E,\ a \neq 0$. Given $t \in T$, we denote by $\gamma_t E$ the closure of the set of all functions of the form $\gamma_t a,\ a \in E$, lying in H. All such subspaces form a family which we will denote by $G_U = \{\gamma_t E \mid t \in T_U\}$. The problem we are interested in is: when does the family G_U form an orhogonal basis from subspaces of H, i. e. when does $\gamma_t E \perp \gamma_s E,\ t \neq s$ and $Cl(\cup \gamma_t E,\ t \in T_U) = H$ hold? (Cl=closure).

We begin with some lemmas.

Lemma 1. *The mapping $f \to f(w)$ is a bounded operator from H_E^2 to E for every $w \in D$.*

Proof. The statement follows from the inequality

$$\|f(w)\|_E = \sup\{|\ (f, k_w a) :\ a \in E,\ \|a\| \leq 1\} \leq \|f\|(k_w(w))^{1/2},\ f \in H_E^2,\ w \in D.$$

Note that it follows by lemma 1. that if the operator $I - \theta(z)U^*$ has a bounded inverse for at least one $z \in D$ then every function in $\gamma_t E$ has the form $\gamma_t a,\ a \in E$.

Lemma 2. *Let H_1 and H_2 be Hilbert spaces with (scalar) reproducing kernels [**4**], $K_w^1(z)$ and $K_w^2(z),\ w \in D,\ z \in D$. If there exists a function h (from D into C) such that $h(z) \neq 0,\ z \in D$, and $K_w^2(z) = \overline{h(w)}h(z)K_w^1(z),\ w \in D,\ z \in D$, then multiplication by h is an isomorphism of spaces H_1 and H_1.*

Proof. We establish the equality

$$(hf, hg)_2 = (f, g)_1,\ f \in H_1,\ g \in H_1, \tag{3}$$

first in the case when $f = \overline{h(w)}K_w^1,\ w \in D$, and $g = \overline{h(\nu)}K_\nu^1,\ \nu \in D:\ (hf, hg)_2 = K_w^2(\nu) = (f, g)_1$. By linearity it follows that (3) hodts also when f and g are linear combinations of functions of the form $\overline{h(w)}K_w^1,\ w \in D$. The same conclusion follows by continuity of the inner product and by completeness of the set $\{\overline{h(w)}K_w^1 \mid w \in D\}$ in H_1 also when f and g are arbitrary functions in H_1. Thus multiplication by h preserves the inner product. Since the set $\{K_w^2 \mid w \in D\}$ is complete in H_2, $hH_1 = H_2$, i. e. multiplication by h is an isomorphism of spaces H_1 and H_2.

Lemma 3. *Let θ be a scalar inner function and $t \in T$. Then the following are equivalent*

(a) *$\gamma_t \in H$ for some complex number U of modulus 1,*

(b) *the limit* $\lim_{r\to 1-} K_{rt}$ *exists in the* H*-norm,*

(c) $\|K_{rt}\|$ *is bounded for* $r<1$.

Proof. $(a)\Rightarrow(b)$. If $\gamma_t\in H$ for some $U, \mid U\mid = 1$, then every function $f\in H$ has a nontangential limit $f(t)$ at t and the functional $f\to f(t)$ is bounded [**3**]. By the existence of the limit $\lim_{r\to 1-}\gamma_t(rt)=\lim_{r\to 1-}(1\theta(rt)\overline{U})(1-r)^{-1}$ it follows that $\lim_{r\to 1-}\theta(rt)=U$ and that $\lim_{r\to 1-}K_w(rt)=\overline{\gamma_t(w)}=(K_w,\gamma_t)$, $w\in D$. This means that $K_w(t)=(K_w,\gamma_t)$, $w\in D$, so $f(t)=(f,\gamma_t)$ for every $f\in H$. In particular

$$\lim_{r\to 1-}(1-\theta(rt)\overline{U})(1-r)^{-1}=\lim_{r\to 1-}\gamma_t(rt)=\|\gamma_t\|^2.$$

This implies that

$$K_{rt}(rt)=(1-\theta(rt)\overline{U})(1-r^2)^{-1}+\theta(rt)\overline{U}(1-\theta\overline{(rt)}U)(1-r^2)^{-1}$$

tends to $\|\gamma_t\|^2$ as $r\to 1-$.

Thus $\|K_{rt}-\gamma_t\|^2=K_{rt}(rt)-\gamma_t(rt)-\overline{\gamma_t(rt)}+\|\gamma_t\|^2\to 0$, as $r\to 1-$, i. e. (b) holds.

$(b)\Rightarrow(c)$ is clear.

$(c)\Rightarrow(a)$. Let θ have the representation $\theta(z)=\nu B(z)S(z)$, $z\in D$, where $\mid\nu\mid=1$,

$$B(z)=\prod_{k=1}^{l}b_k(z)=\prod_{k=1}^{l}\mid z_k\mid/z_k(z_k-z)(1-z\overline{z_k})^{-1},$$

$z\in D$, with $z_k\in D$ for $k=1,2,\dots,\ l$ $(1\le l\le\infty;\ \mid z_k\mid/z_k=1$, if $z_k=0)$ (if θ has no zeros then $B(z)\equiv 1$), and

$$S(z)=\exp\Big(-\int_0^{2\pi}(s+z)(s-z)^{-1}d\mu(x)\Big),\ z\in D,\ (s=e^{ix}),$$

where μ is a finite, non-negative singular measure on T. From boundedness of $\|K_{rt}\|^2=K_{rt}(rt)$ and from $\mid B(rt)\mid\ge\mid\theta(rt)\mid$ and $\mid S(rt)\mid\ge\theta(rt)\mid$ it follows that $(1-\mid B(rt)\mid^2)(1-r^2)^{-1}$ and $(1-\mid S(rt)\mid^2)(1-r^2)^{-1}$ are bounded. Since

$$\begin{aligned}(1-\mid B(rt)\mid^2)(1-r^2)^{-1}&=(1-\mid z_1\mid^2)\mid 1-rt\overline{z_1}\mid^{-2}+\\&+\sum_{k=1}^{l}\prod_{j=1}^{k-1}\mid b_j(rt)\mid^2(1-\mid z_k\mid^2)\mid 1-rt\overline{z_k}\mid^{-2}\to\\&\to\sum_{k=1}^{l}(1-\mid z_k\mid^2)\mid 1-\overline{t}z_k\mid^{-2}\quad\text{as } r\to 1-,\end{aligned}$$

it follows that

$$\sum_{k=1}^{l}(1-\mid z_k\mid^2)\mid 1-t\overline{z_k}\mid^{-2}<\infty. \tag{4}$$

Since $| S(rt) |^2 = \exp\Big(-2(1-r^2)\int_0^{2\pi} | s-rt |^{-2} \, d\mu(x)\Big)$, it follows from boundednes of $(1- | S(rt) |^2 \, (1-r^2)^{-1}$ that $\int_0^{2\pi} | s-rt |^{-2} \, d\mu(x)$ is bounded for r sufficiently near to 1, which gives

$$\int_0^{2\pi} | s-t |^{-2} \, d\mu(x) < \infty. \tag{5}$$

Now, (4) and (5) impply that $\gamma_t \in H$ for some U, $| U |= 1$, [**3**]. This completes the proof.

Remark 1. Let $\Re\varphi(rt)(1-r)^{-1}$ be bounded, $t \in T(\varphi = \varphi_1)$. Then $\|K_{rt}\|$ is bounded also. This is evident from the relation

$$K_{rt}(rt) = \Re\varphi(rt)(1-r^2)^{-1} \, | \, 1-\theta(rt) \, |^2 \, .$$

LEMMA 4. *Let the operator $I-\theta(z)$ have a bounded inverse for every $z \in D$ and let $\Re\varphi(rt) \to 0$, $r \to 1-$, $(\varphi = \varphi_I)$ (at least in the weak operator convergence) for a. e. $t \in T$. Fix $a \in E\backslash\{0\}$ and put $\varphi_a(z) = (\varphi(z)a, \, a)_E$, $z \in D$. Then the function $\theta_a(\varphi_a - 1)(\varphi_a + 1)^{-1}$ is a (scalar) inner function and the corresponding space $H_a = H(\theta_a)$ is isometrically isomorphic to the subspace Ka of H generated by functions of the form $K_w(z)(I-\theta(w)^*)^{-1}a$, $w \in D$. An isomorphism Φ from Ka to H_a is given by $\Phi f(z) = (1-\theta_a(z))((I-\theta(z))^{-1}f(z), \, a)_E$, $z \in D$, $f \in Ka$.*

Proof. Since $\|\theta(z)\| \le 1$, $z \in D$, and

$$\Re\varphi(z) = (I-\theta(z))^{-1}(I-\theta(z)^*\theta(z))(I-\theta(z)^*)^{-1},$$

it follows that $\Re\varphi(z) \ge 0$ and $\Re\varphi_\alpha(z) \ge 0$, which implies $| \, \theta_a \, | \le 1$, $z \in D$. Since $\Re\varphi(rt) \to 0$, $r \to 1-$, for a. e. $t \in T$, it follows that the same holds for $\Re\varphi_a$ and so radial limits of θ_a have modulus 1 for a. e. $t \in T$. Thus θ_a is an inner function.

Now consider the mapping Φ_1 defined by $\Phi_1 f(z) = ((I-\theta(z))^{-1}f(z), \, a)_E$, $z \in D$, $f \in Ka$. Because of $\Phi_a f(z) = (f, K_Z(I-\theta(z)^*)^{-1}a) \; \Phi_1$ is a regular mapping, i.e. $\Phi_1 f = 0$ iff $f = 0$. So Φ_1 maps Ka one-to-one onto a set $L = L_a$ of scalar analytic (in D) functions. If we define in L the inner product by $(h_1, h_2)_L = (\Phi_1^{-1}h_1, \, \Phi_1^{-1}h_2)$, h_1, $h_2 \in L$, then L becomes a Hilbert space isometrically isomorphism to Ka. The space L possesses also reproducing kernel. This is the function

$$J_w(z) = \Phi_1 K_w(z)(I-\theta(W)^*)^{-1}a = (\varphi_a(z) + \overline{\varphi_a(w)})2^{-1}(1-z\overline{w})^{-1}, \; z \in D, \; w \in D.$$

Finally, multiplication by the function $1-\theta_a(z)$ is an isometrical isomorphism from L onto H_a (Lemma 2). Thus Φ is really an isometrical isomorphism from Ka onto H_a.

LEMMA 5. *Let the assumptions of Lemma 4 be satisfied and let $t \in T_I$. Then there exists an operator $\alpha(t) \in B(E,E)$ such that*

$$\lim_{r \to 1-} (1-r)(I-\theta(rt)^*)^{-1} = \alpha(t) \tag{6}$$

in the strong operator convergence. If $a \in A\backslash\{0\}$, then the function $\gamma_t(\alpha(t)a(\gamma_t(z) = \gamma_t(t,z)$, see (2)) belongs to the subspace Ka (defined in Lemma 4) and it holds

$$(7) \qquad \lim_{r\to 1-}(1-r)K_{rt}(I-\theta(rt)^*)^{-1}a = \gamma_t\alpha(t)a$$

in the H-norm. If θ_a, H_a and Φ are as in Lemma 4 and if γ_t^a denotes the function $(1-\theta_a))(1-z\overline{t})^{-1}$, then $\gamma_t^a \in N_a$ iff $(\alpha(t)a,\ a)_E \neq 0$ and it is also

$$(8) \qquad \Phi\gamma_t\alpha(t)a = (\alpha(t)a,\ a)_E\gamma_t^a.$$

Proof. Since $t \in T_I$, it follows that $\gamma_t b \in H$ for some $b \in E\backslash\{0\}$. Let $a \in E$ and $(b,a)_E \neq 0$. Denote by $P\gamma_t b$ the projection of $\gamma_t b$ to the subspace Ka. Because of

$$((I-\theta(z))^{-1}P\gamma_t(z)b, a)_E = (\gamma_t b,\ K_z(I-\theta(z)^*)^{-1}a) = (b,a)_E(1-z\overline{t})^{-1},\ z \in D,$$

it must be

$$(9) \qquad \Phi P\gamma_t b = (b,a)_E\gamma_t^a.$$

Since $(b,a)_E \neq 0$, the function γ_t^a lies in H_a. If $K_w^a(z)$ denotes the reproducing kernel in H_a, then by Lemma 3 $\gamma_t^a = \lim_{r\to 1-}\underline{K_{rt}^a}$ in the H_a-norm. Since Φ is an isomorphism (Lemma 4) and $\Phi^{-1}K_{rt}^a = (1-\overline{\theta_a(rt)})K_{rt}(I-\theta(rt)^*)^{-1}a$ we have also

$$\Phi^{-1}\gamma_t^a = \lim_{r\to 1-}(1-\overline{\theta_a(rt)})K_{rt}(I-\theta(rt)^*)^{-1}a$$

in the H-norm. Regarding the fact that

$$\lim_{r\to 1-}(1-\theta_a(rt))(1-r)^{-1} = \lim_{r\to 1-}(\gamma_t^a, K_{rt}^a)_{H_a} = \|\gamma_t^a\|^2,$$

we obtain

$$(10) \qquad \Phi^{-1}\gamma_t^a = \|\gamma_t^a\|^2\lim_{r\to 1-}(1-r)K_{rt}(I-\theta(rt)^*)^{-1}a.$$

If we consider pointwise convergence (Lemma 1) in the last relation, we can conclude that there exists

$$(11) \qquad \lim_{r\to 1-}(1-r)(I-\theta(rt)^*)^{-1}a \stackrel{\text{def}}{=} \alpha(t)a$$

in the E-norm and that (7) must hold, which gives also $\gamma_t\alpha(t)a \in Ka$. In fact, the limit (11) exists and the relation (7) holds for every $a \in E$, for if $(b,a)_E = 0$ we can write $a = (a+b)-b$. Since a in (11) may be arbitrary, $\alpha(t)$ is a (bounded) operator and (6) follows. Putting $b = \alpha(t)a$ in (9) we obtain (8). Now (10) and (7) imply $\Phi^{-1}\gamma_t^a = \|\gamma_t^a\|^a\gamma_t\alpha(t)a$. Comparing this with (8) we see that $(\alpha(t)a,\ a)_E = \|\gamma_t^a\|^{-2}$. Hence it is evident that $\gamma_t^a \in H_a$ implies $(\alpha(t)a,\ a)_E \neq 0$ and (8) shows that the converse is also true.

Lemma 6. *In Lemma 5 all functions of the form $\gamma_t\alpha(t)a$, $a \in E$, form a complete set in $\gamma_t E$.*

Proof. If $\gamma_t b \in \gamma_t E$ and $\gamma b \perp \gamma_t \alpha(t)a,\ a \in E$, then by (7) $0 = (\gamma_t b, \gamma_t \alpha(t)a) = \lim\limits_{r\to 1-}(1-r)(\gamma_t b, K_{rt}(I-\theta(rt)^*)^{-1}a) = (b,a)_E,\ a \in E$, i. e. $b = 0$ and $\gamma_t b = 0$.

LEMMA 7. *Let the assumptions of Lemma* 4 *be satisfied. Then the set* $G = G_I$ *is orthogonal.*

Proof. Let $t \in T_l,\ s \in T_I,\ t \neq s$, and let $\gamma_t\alpha(t)a \in \gamma_t E$ and $\gamma_s b \in \gamma_s E$. Then it follows by (7) that

$$(\gamma_t\alpha(t)a, \gamma_s b) = \lim_{r\to 1-}(1-r)(1-r\bar{t}s)^{-1}(a,b)_E = 0.$$

By completeness of the set $\{\gamma_t\alpha(t)a \mid a \in E\}$ in $\gamma_t E$ it follows that $\gamma_t E \perp \gamma_s E$. Thus the family G is orthogonal.

THEOREM. *Let* θ *be an inner operator function,* U *a unitary operator in* E *and let the operator* $I - \theta(z)U^*$ *have a bounded inverse for every* $z \in D$. *If* $(1-r)^{-1}\Re\varphi(rt)$ *is bounded in* r *for all* $t \in T$ *except for a countable set, then the family* G_U *is orthogonal and complete in* H.

Proof. Since $H(\theta U^*) = H(\theta)$ for each unitary operator U (*in* E), it is enough to give the proof only in the case $U = I$. Thus let $U = I$. The assumption on boundedness of $(1-r)^{-1}\Re\varphi(rt)$ implies that $\lim_{r\to 1-}\Re\varphi(rt) = 0$ in the strong operator convergence for all $t \in T$ except for a countable set. So the assumptions of Lemmas 4, 5, 6, 7 are satisfied.

Orthogonality of the family G is proved in Lemma 7. Let us prove the completeness of G. It is clear that whenever $(1-r)^{-1}\Re\varphi(rt)$ is bounded then $(1-r)^{-1}\Re\varphi_a(rt)$ is too, for $a \in E$ (φ_a as in Lemma 4). By Remark 1 and by Lemma 3 it follows that the condition (a) in Lemma 3 is satisfied for all $t \in T$ except for a countable set. By Theorem 7.1 and Lemma 3.1 in [**3**] it follows that the set of functions of the form $\gamma_t^a,\ t \in T$, which belong to H_a is complete in H_a. By Lemma 5 (relation (8)), Φ maps the set of all functions of the form $\nu\gamma_t\alpha(t)a,\ t \in T_I,\ \nu \in C$ (a fixed), onto the set of all functions of the form $\nu\gamma_t^a,\ t \in T,\ \nu \in C$, which belong to H_a. This implies that the set of functions of the form $\gamma_t\alpha(t)a,\ t \in T_I$, is complete Ka. If a function f in H is orthogonal to all subspaces of the type $\gamma_t E,\ t \in T_I$, it is orthogonal also to all functions of the form $\gamma_t\alpha(t)a,\ t \in T_I,\ a \in E$. Since the above set of functions for fixed a is complete in Ka, that implies $f \perp Ka$ for every $a \in E$. However, this implies that $((I-\theta(w))^{-1}f(w),\ a)_E = (f, K_w(I-\theta(w)^*)^{-1}a) = 0$ for every $a \in E$ and every $w \in D$, so that $f = 0$. Thus, the set G is complete in H. This completes the proof.

Remark 2. If the function θ admits analytic continuation across some point $t \in T$ and if $\theta(t) = U$, then $\gamma_t a \in H$ for every $a \in E$ and $\gamma_t(z)$ is obtained by evaluation of the (analytically continued) reproducing kernel $K_w(z)$ for $w = t$. In the general case the situation is, in a sense, similar. Namely, it follows easily by (7) that, for $t \in T_I,\ a \in E$ and $z \in D$, $\lim_{r\to 1-} K_{rt}(z)\alpha(t)a = \gamma_t(z)\alpha(t)a$ in the E-norm. With the help of the last relation $K_w(z)$ can be extended for every $t \in T_I$ along the radius $\{rt \mid 0 \leq r \leq q\}$ at least as an operator function with values in the set of bounded operators from $\alpha(t)E$ into $\alpha(t)E$, so that we can consider $\gamma_t(z)$ also in the general case as an evaluation of $K_w(z)$ for $w = t$.

REFERENCES

[1] H. Helson, *Lectures on Invariant Subspaces*, Academic Press, New York, 1964.

[2] B. Sz.-Nagy, C. Foiaş, *Analyse harmonique des opérateurs de l'espace de Hilbert*, Masson-Akad. Kiadó, Paris et Budapest 1967.

[3] D. N. Klark, *One dimensional perturbations of restriced shifts*, J. Anal. Math. **25** (1972), 169–191.

[4] N. Aronszajn,*Theory of reproducing kernels*, Trans. Amer. Math. Soc. **68** (1950), 337–404.

Mašinski fakultet
Univerzitet u Beogradu
11000 Beograd

(Received 05 05 1983)