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ON THE ABSOLUTE SUMMABILITY OF LACUNARY FOURIER SERIES

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Abstract. Let $f \in L[-\pi,\pi]$ and let its Foirer Series $\sigma(f)$ be lacynary. The absolute convergence of $\sigma(f)$ when f satisfies Lipschitz condition of order α, $0<\alpha<1$, only at a point and when $\{n_k\}$ satisfies the gap condition $n_{k+1}-n_k \geq An_K^\beta k^\gamma$ $(0<\beta<1,\ \gamma\geq 0)$ is obtained by Patadian and Shah when $\alpha\beta+\alpha\gamma>(1-\beta)/2$. Here we study the absolute summability of $\sigma(f)$ when $\alpha\beta+\alpha\gamma\leq(1-\beta)/2$.

1. Let

$$\sum_{k=1}^{\infty}(a_{n_k}\cos n_kx+b_{nk}\sin n_kx) \tag{1.1}$$

be the Fourier series of a 2π-periodic function $f\in L[-\pi,\pi]$ with an infinity of gaps (n_k,n_{k+1}), where $\{n_k\}$ $(k\in N)$ is a strictly increasing sequence of natural numbers. Noble [**7**], Kennedy [**4, 5, 6**], and several other mathemaicians , have studied the absolute convergence of the Fourier series (1.1), as well as the order of magnitude of Fourier coefficients, by considering various properties of f either on an arbitrary subinterval or on an arbitrary subset of $[-\pi,\pi]$ of positive measure. This way they obtained a number of results under different lacunarity conditions. Izumi and Izumi [**3**], Chao [**1**], and Patadia and Shah [**8**], have studied this problem for the Fourier series (1.1) with some lacunae when the function satisfies Lipschitz condition only at a point. Chao [**1**] proved the following theorems:

THEOREM A. [**1**; Theorem 1]. *If*

$$\text{(i)}\quad f\in \operatorname{Lip}\alpha\,(\alpha>0)\ \text{ at a point } x_0\in(-\pi,\pi), \tag{1.2}$$

$$\text{(ii)}\quad n_{k+1}-n_k\geq A\ F(n_k) \tag{1.3}$$

where $F(n_k)\uparrow\infty$ *as* $k\to\infty$, $F(n_k)\leq n_k$ *for all* k *and* A *is a positive constant, then*

$$a_{n_k},b_{n_k}=O(F(n_k)^{-\alpha}),\ k=1,2,\dots \tag{1.4}$$

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THEOREM B. [**1**; Theorem 2]. *If f satisfies* (1.2) *and if*

$$n_{k+1} - n_k \geq An_k^{\beta} k^{\gamma} \ (0 < \beta < 1,\ \ \gamma \geq 0) \tag{1.5}$$

where A is a positive constant, then the Fourier series (1.1) *of f converges absolutely when $\alpha\beta + \alpha\gamma + \beta > 1$.*

Furthermore, Patadia and Shah [**8**] considered the same gap condition (1.5) and proved the following theorem:

THEOREM C. *If f satisfies* (1.2), *and if $\{n_k\}$ satisfies* (1.5), *then*

$$\sum_{k=1}^{\infty}(\mid a_{n_k} \mid^r + \mid b_{n_k} \mid^r) < \infty \ \ 0 < r \leq 1 \tag{1.6}$$

when $\alpha\beta r + \alpha r\gamma > (1 - r/2)(1 - \beta)$.

We observe that the particular case of theorem C when $r = 1$ provides us with a generalization of Theorem B, ensuring the absolute convergence o the Fourier series (1.1) when $\alpha\beta + \alpha\gamma > (1 - \beta)/2$. It may be noted here that when $\alpha\beta + \alpha\gamma = (1 - \beta)/2$, the absolute convergence of (1.1) is obtained by Patadie and Shah [**9**] by taking at apoint a little stronger condition than $\mathrm{Lip}\,\alpha$ on f.

Now, it is quite natural to inequire into the behaviour of the Fourier series (1.1) of a function f in $\mathrm{Lip}\,\alpha$ at a point, when $\alpha\beta + \alpha\gamma \leq (1 - \beta)/2$. In this regard, we propose to study the absolute summability (c, θ) of the series (1.1). We prove the following theorem:

THEOREM. *If $f \in Lip\,\alpha(0 < \alpha < 1)$ at a point $x_0 \in (-\pi, \pi)$, and if $\{n_k\}$ satisfies* (1.5) *with some suitable constant A, then the Fourier series* (1.1) *of f is absolutely summable (c, θ) for $0 < \theta \leq 1$ when*

$$\alpha > \max\left\{\frac{1 - \beta - \theta - \gamma\theta}{\beta + \gamma}, \frac{2 - 3\beta - \gamma + \beta\theta - \theta}{\beta + \beta\gamma}\right\}.$$

Remark 1. Theorems 1 and 2 due to Patel [**10**] are particular cases of this theorem when $\theta = 1,\ \gamma = 0$, and $\theta = 1/2,\ \gamma = 0$ respectively.

Remark 2. It is interesting to observe that when $\gamma = 1$, the theorem gives the absolute summability $(c, 1)$ of the Fourier series (1.1) for every $\alpha > 0$; and that, when $\gamma = 3/2$, we get the absolute summability $(c, 1/2)$ of (1.1) for every $\alpha > 0$

2. We need the following lemma due to Patadia and Shah [**9**].

LEMMA. *If $\{n_k\}$ satisfies* (1.5) *with $A > 2^M - 1$, M being a positive integer reater than, δ , where $\delta = (1 + \gamma)/(1 - \beta)$, then*

$$n_k \geq k^{\delta} \ \ for\ all\ \ k \in N \tag{2.1}$$

Proof of the Theorem. For a real number s, which is not a negative integer, put $E_n^s = \binom{n+s}{n}$ where $n \in N$ and $E_0^s = 1$. Denoting the n-th Cesaro mean of order $\theta > 0$ by $\sigma_n^\theta(x)$, and replacing the absent terms in (1.1) by zeros, we have [**2**]:

$$
\begin{aligned}
\mid \sigma_{n_k}^\theta(x) - \sigma_{n_k-1}^\theta(x) \mid &= \frac{1}{n_k \cdot E_{n_k}^\theta} \Big| \sum_{p=1}^{k} E_{n_k-n_p}^{\theta-1} \cdot n_p \cdot (a_{n_p} \cos n_p x + b_{n_p} \sin n_p x) \Big| \\
&\le \frac{1}{n_k \cdot E_{n_k}^\theta} \Big\{ \mid n_k (a_{n_k} \cos n_k x + b_{n_k} \sin n_k x) \mid + \\
&+ \Big| \sum_{p=1}^{k-1} E_{n_k-n_p}^{\theta-1} \cdot n_p \cdot (a_{n_p}) \cos n_p x + b_{n_p} \sin n_p x) \Big| \Big\}. \qquad (2.2)
\end{aligned}
$$

Let $0 < \theta \le 1$. Now.

$$
\text{(i)} \quad E_n^\theta \simeq \frac{n^\theta}{\Gamma(\theta+1)}, \quad \text{(ii)} \;\; a_{n_k},\ b_{n_k} = O\Big(\frac{1}{n_k^{\alpha\beta} \cdot k^{\gamma\alpha}}\Big), \;\; k = 1, 2, 3, \dots,
$$

by taking $F(n_k) = n_k^\beta k^\gamma$ in Theorem A, and

$$
\begin{aligned}
\text{(iii)} \quad \mid n_k - n_p \mid \ge \mid n_k - n_{k-1} \mid \quad &\text{for } p = 1, 2, 3, \dots, k-1 \\
\ge A n_k^\beta k^\gamma, \;\; &\text{by (1.5).}
\end{aligned}
$$

Hence, from (2.1) and (2.2), we obtain

$$
\begin{aligned}
&\mid \sigma_{n_k}^\theta(x) - \sigma_{n_k-1}^\theta(x) \mid \\
&= 0(1) \frac{1}{n_k n_k^\theta} \Big\{ n_k n_k^{-\alpha\beta} k^{-\gamma\alpha} + \sum_{p=1}^{k-1} \frac{1}{(n_k - n_p)^{1-\theta}} \cdot n_p \cdot n_p^{-\alpha\beta} p^{-\gamma\alpha} \Big\} \\
&= 0(1) \frac{1}{n_k^{1+\theta}} \Big\{ n_k^{1-\alpha\beta} k^{-\gamma\alpha} + \Big(\frac{1}{n_k^\beta k^\gamma}\Big)^{1-\theta} \sum_{p=1}^{k-1} \frac{n_p^{1-\alpha\beta}}{p^{\gamma\alpha}} \Big\} \\
&= 0(1) \frac{1}{n_k^{1+\theta}} \Big\{ n_k^{1-\alpha\beta} k^{-\gamma\alpha} + \Big(\frac{1}{n_k^\beta k^\gamma}\Big)^{1-\theta} \cdot k \cdot n_k^{1-\alpha\beta} \Big\},
\end{aligned}
$$

as $p^{-\gamma\alpha} \le 1$ and $n_p^{1-\alpha\beta} \le n_k^{1-\alpha\beta}$, $0 < \alpha, \beta < 1$. Therefore

$$
\begin{aligned}
\mid \sigma_{n_k}^\theta(x) - \sigma_{n_k-1}^\theta(x) \mid &= 0(1) \Big\{ \frac{1}{n_k^{\theta+\alpha\beta} k^{\gamma\alpha}} + \frac{1}{n_k^{\theta+\beta-\beta\theta+\alpha\beta} k^{\gamma-\gamma\theta-1}} \Big\} \\
&= 0(1) \Big\{ \frac{1}{k^{\delta(\theta+\alpha\beta)+\gamma\alpha}} + \frac{1}{k^{\delta(\theta+\beta-\beta\theta+\alpha\beta)+\gamma-\gamma\theta-1}} \Big\} \\
&= 0(1) \Big\{ \exp_k \Big(\frac{\theta + \alpha\beta + \gamma\theta + \alpha\gamma}{1-\beta} \Big) + \\
&+ \exp_k \Big(\frac{\theta + 2\beta - \beta\theta + \alpha\beta\gamma\alpha\beta + \gamma - 1}{1-\beta} \Big) \Big\}, \qquad (2.3)
\end{aligned}
$$

as $\delta = (1+\gamma)/(1-\beta)$, $(\exp_k A$ denotes $k^{-A})$. Finally, since $\alpha > (1-\beta-\theta\gamma\theta)/(\beta+\gamma)$, it follows that $(\theta + \alpha\beta + \gamma\theta + \alpha\gamma)/(1-\beta) > 1$; and since

$$\alpha > \frac{2 - 3\beta - \gamma + \beta\theta - \theta}{\beta + \beta\gamma}$$

we have

$$\frac{\theta + 2\beta - \beta\theta + \alpha\beta + \alpha\beta\gamma + \gamma - 1}{1-\beta} > 1.$$

Hence, from (2.3) we have

$$\sum_{k=1}^{\infty} \mid \sigma_{n_k}^{\theta}(x) - \sigma_{n_k-1}^{\theta}(x) \mid < \infty,$$

which implies the absolute summability (c, θ) of (1.1). This completes the proof of the theorem.

REFERENCES

[1] Jia-arng Chao, *On Fourier series with gaps*, Proc. Japan Acad. **42** (1966), 308–312.

[2] T. M. Flett, *Some remarks on strong summability*, Quart. J. Math. (Oxford) **10** (1959), 115–139.

[3] M. Izumi, S. I. Izumi, *On lacunary Fourier series*, Proc. Japan Acad. **41** (1965), 648–651.

[4] P. B. Kennedy, *Fourier series with gaps*, Quadt. J. Math. (Oxford) (2) **7** (1956), 224– 230.

[5] P. B. Kennedy, *On the coefficients of certain Fourier series*, J. London Math. Soc. **33** (1958), 196–207.

[6] P. B. Kennedy, *Note on Fourier series with Hadamard gaps*, J. London Math. Soc. **39** (1964), 115–116.

[7] M. E. Noble, *Coefficient properties of Fourier series with a gap condition*, Math. Annalen **128** (1954), 55–62.

[8] J. R. Patadia, V. M. Shah, *On the absolute convergence of lacunary Fourier series*, Proc. Amer. Math. Soc. **83** (1981), 680–682.

[9] J. R. Patadia, V. M. Shah, *On the absolute convergence of lacunary Fourier series*, to appear

[10] N. V. Patel, *On the absolute summability of a lacunary Fourier series*, J. M. S. University of Baroda, Volume XXIX, Science No. 3 (1980), 49–56.

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