

ON APPROXIMATION BY MODIFIED BERNSTEIN POLYNOMIALS

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Abstract. We study some theorems on the approximation of the r -th derivative of a given function f by corresponding r -th derivative of the modified Bernstein polynomials.

1. Introduction. For a function f defined on the closed interval $[0, 1]$ of the real x -axis, the expression

$$(1.1) \quad (B_n f)(x) = \sum_{k=0}^n p_{n,k}(x) f(k/n),$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, is called the Bernstein polynomial of order n of the function f .

Recently Derriennic [1] studied a new kind of positive linear operators $\{L_n\}$ for a Lebesgue integrable function $f \in L^1[0, 1]$, defined as

$$(1.2) \quad (L_n f)(x) = (n+1) \sum_{k=0}^n p_{n,k}(x) \int_0^1 p_{n,k}(t) f(t) dt,$$

which is a modification of the polynomials (1.1).

Following [1], for $r \leq n$, we get

$$(1.3) \quad (L_n^{(r)} f)(x) = \frac{(n+1)!n!}{(n-r)!(n+r)!} \sum^*) p_{n-r,k}(x) \int_0^1 p_{n+r,k+r}(t) \frac{d^r f(t)}{dt^r} dt$$

We will prove some theorems on the approximation of the r -th derivative of a function f by the corresponding operators $(L_n^{(r)})$.

2. In this section we prove the following lemma which will be useful in proving the theorems.

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^{*)} \sum stands for $\sum_{k=0}^{n-r}$.

LEMMA 2.1. For $r \leq n$, let

$$(2.1) \quad T_{n-r,m-r}(x) = \frac{m!}{(m-r)!} \sum p_{n-r,k}(x) \int_0^1 p_{n+r,k+r}(t) (t-x)^{m-r} dt.$$

Then one has the following relation

$$(2.2) \quad \begin{aligned} (n+m+2) \left(\frac{m-r+1}{m+1} \right) T_{n-r,m-r+1}(x) \\ = (m+1)(1-2x)T_{n-r,m-r}(x) + 2mx(1-x)T_{n-r,m-r-1}(x) + \\ + x(1-x)T'_{n-r,m-r}(x), \end{aligned}$$

with $T_{n-r,0} = \frac{r!}{(n+r+1)}$. In particular we get

$$(2.3) \quad \frac{1}{(r+1)!} T_{n-r,1}(x) = \frac{(r+1)(1-2x)}{(n+r+2)(n+r+1)}$$

$$(2.4) \quad \begin{aligned} \frac{2}{(r+2)!} T_{n-r,2}(x) = \\ = \frac{(n+1)}{(n+r+3)(n+r+2)(n+r+1)} \left\{ \frac{(r+1)(r+2)(1-2x)^2}{n+1} + 2x(1-x) \right\}. \end{aligned}$$

Proof. Differentiating (2.1) w. r. t. x we get

$$\begin{aligned} x(1-x)T'_{n-r,m-r}(x) &= \frac{m!}{(m-r)!} \sum_{k=0}^{n-r} p'_{n-r,k}(x) x(1-x) \int_0^1 p_{n+r,k+r}(t) \cdots \\ &\quad \cdots (t-x)^{m-r} dt - mx(1-x)T_{n-r,m-r-1}(x). \end{aligned}$$

Using the relation $x(1-x)p'_{n-r,k}(x) = \{k - (n-r)x\}p_{n-r,k}(x)$, we get after simplification

$$\begin{aligned} x(1-x)T'_{n-r,m-r}(x) &= \frac{m!}{(m-r)!} \sum p_{n-r,k}(x) \int_0^1 \{k - (n-r)t + (n-r)(t-x)\} \cdots \\ &\quad \cdots p_{n+r,k+r}(t) (t-x)^{m-r} dt - mx(1-x)T_{n-r,m-r-1}(x), \\ &= \frac{m!}{(m-r)!} \sum p_{n-r,k}(x) \int_0^1 t(1-t)p'_{n+r,k+r}(t) (t-x)^{m-r} dt \cdots \\ &\quad \cdots - r(1-2x)T_{n-r,m-r}(x) + \left(\frac{m-r+1}{m+1} \right) (n+r)T_{n-r,m-r+1}(x) - \\ &\quad - mx(1-x)T_{n-r,m-r-1}(x). \end{aligned}$$

Again using $t(1-t) = -(t-x)^2 + (1-2x)(t-x) + x(1-x)$ and integrating the r. h. s. by parts, we get

$$\begin{aligned} x(1-x)T'_{n-r,m-r}(x) &= (m-r+2) \left(\frac{m-r+1}{m+1} \right) T_{n-r,m-r+1}(x) - (m-r+1)(1-2x)T_{n-r,m-r}(x) - \\ &\quad - mx(1-x)T_{n-r,m-r-1}(x) - r(1-2x)T_{n-r,m-r}(x) + \\ &\quad + \left(\frac{m-r+1}{m+1} \right) (n+r)T_{n-r,m-r+1}(x) - mx(1-x)T_{n-r,m-r-1}(x) \end{aligned}$$

which on further simplification leads to the required result (2.2).

PROPOSITION 2.2. For $r \leq n$, we have

$$(2.5) \quad L_n^{(r)}(t-x)(x) = \frac{n!(n+1)!}{(n-r)!(n-r+2)!}(1-2x)$$

$$(2.6) \quad L_n^{(r)}(t-x)^2(x) = \frac{(n+1)!^2}{(n-r)!(n-r+3)!} \left\{ \frac{(r+1)(r+2)(1-2x)^2}{n+1} + 2x(1-x) \right\}$$

Proof. The proof follows easily from (1.3), (2.3) and (2.4).

PROPOSITION 2.3. For $r \leq n$, one has,

$$(2.7) \quad L_n^{(r)}(t-x)^2(x) \leq (r+2)/2(n+1), \quad (0 \leq x \leq 1).$$

Proof. We can easily show from (2.6) that,

$$\begin{aligned} L_n^{(r)}(t-x)^2(x) &\leq \frac{1}{n+r+2} \left\{ \frac{(r+2)(r+1)(1-2x)^2}{n+1} + 2x(1-x) \right\} \\ &\leq \frac{(r+2)(r+1)}{(n+1)(n+r+2)} \leq \frac{(r+2)}{2(n+1)}, \quad r \leq n. \end{aligned}$$

3. THEOREM 5.1. If $f^{(r)}$ is bounded and integrable in $[0, 1]$ and admits $(r+2)$ -th derivative at a point $x \in [0, 1]$, then

$$(3.1) \quad \lim_{n \rightarrow \infty} n \{ (L_n^{(r)} f)(x) - f^{(r)}(x) \} = (r+1)(1-2x)f^{(r+1)}(x) + x(1+x)f^{(r+2)}(x).$$

Proof. By the Taylor formula, we have

$$f^{(r)}(t) - f^{(r)}(x) = (t-x)f^{(r+1)}(x) + (t-x)^2/2 \cdot f^{(r+2)}(x) + (t-x)^2/2 \cdot \eta(t-x),$$

where $\eta(u) \rightarrow 0$ as $u \rightarrow 0$ and η is bounded in $[-x, 1-x]$ and integrable. Now, applying (1.3) to this and using the results (2.5) and (2.6), we get

$$(3.2) \quad \begin{aligned} (L_n^{(r)} f)(x) - f^{(r)}(x) &= \left[(r+1)(1-2x)f^{(r+1)}(x) + \left\{ \frac{(r+1)(r+2)(1-2x)^2}{2(n+1)} \right. \right. \\ &\quad \left. \left. + x(1-x) \right\} \frac{n+1}{n+r+3} f^{(r+2)}(x) \right] \frac{n!(n+1)!}{(n-r)!(n+r+2)!} + E_{n,e}(x) \end{aligned}$$

$$\text{where } E_{n,r}(x) = \frac{(n+1)!n!}{2(n-r)!(n+r)!} \sum p_{n-r,k}(x) \int_0^1 p_{n+r,k+r}(t)(t-x)^2 \eta(t-x) dt.$$

We shall show that $nE_{n,r}(x) \rightarrow 0$ as $n \rightarrow \infty$. Let $M = \sup_{u \in [-x, 1-x]} |\eta(u)|$ and let $\varepsilon < 0$ be arbitrary. Choose $\delta > 0$ such that $|\eta(u)| < \varepsilon$ when $|u| \leq \delta$. So for all $t \in [0, 1]$, we have $|\eta(t-x)| < \varepsilon + M(t-x)^2/\delta^2$. Clearly

$$\begin{aligned} nE_{n,r}(x) &< \varepsilon n/2 \cdot L_n^{(r)}(t-x)^2(x) + Mn/2\delta^2 \cdot L_n^{(r)}(t-x)^4(x) \\ &= \frac{\varepsilon n}{2} \frac{(n+1)!(n+1)!}{(n-r)!(n+r+3)!} \left\{ \frac{(r+1)(r+2)(1-2x)^2}{n+1} + 2x(1-x) \right\} + \frac{M}{2\delta^2} O\left(\frac{1}{n}\right); \end{aligned}$$

which show that $nE_{n,r}(x) \rightarrow 0$ as $n \rightarrow \infty$. Thus, as $n \rightarrow \infty$, we get the required result from (3.2). This completes the proof.

THEOREM 3.2. *Let $f \in C^{(r+1)}[0, 1]$ and let $w(f^{(r+1)}; \cdot)$ be the moduli of continuity of $f^{(r+1)}$. Then for $n \geq r$, ($r = 0, 1, 2, \dots$), we have*

$$(3.3) \quad \|L_n^{(r)}f - f^{(r)}\| \leq (r+1)/(n+r+2)\|f^{(r+1)}\| + 1/\sqrt{n}\{\sqrt{\lambda r} + \lambda r/2\} \cdots \\ \cdots w(f^{(r+1)}; 1/\sqrt{n}),$$

where the norm is sup-norm over $[0, 1]$, and $\lambda r = 1 + r/2$.

Proof. Following [2], we write

$$f^{(r)} - f^{(r)}(x) = (t-x)f^{(r+1)}(x) + \int_x^t \{f^{(r+1)}(y) - f^{(r+1)}(x)\}dy.$$

Now, applying (1.3) to the above and using the inequality

$$|f^{(r+1)}(y) - f^{(r+1)}(x)| \leq \{1 + |y-x|/\delta\}w(f^{(r+1)}; \delta),$$

and the results (2.5) and (2.6), we get

$$\begin{aligned} & |(L_n^{(r)}f)(x) - f^{(r)}(x)| \\ & \leq |f^{(r+1)}(x)|L_n^{(r)}(t-x)(x) + w(f^{(r+1)}; \delta)L_n^{(r)}\left[\left|\int_x^t 1 + |y-x|/\delta dy\right|\right](x), \\ & \leq |f^{(r+1)}(x)|L_n^{(r)}(t-x)(x) + w(f^{(r+1)}; \delta)\left\{\sqrt{L_n^{(r)}(t-x)^2(x)}\right. \\ & \quad \left.+ L_n^{(r)}(t-x)^2(x)/2\delta\right\}. \end{aligned}$$

Choosing $\delta = 1/\sqrt{n}$ and using the result (2.7), we get the required result (3.3). This completes the proof.

Remark. By putting $r = 0$ in (3.1), we get Theorem 2.5 of Derriennic [1].

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