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ON THE MAXIMUM AND MINIMUM CHAIN CONDITIONS FOR THE "LARGENESS" ORDERING ON THE CLASS OF GROUPS

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Abstract. In previous papers the autor has defined a quasi-order $\preceq$ on the class of groups (the"largeness" ordering). One can then define the *height* of group, and also define what it means for a group to satisfy max-$\preceq$ or min-$\preceq$. A natural question is whether the finiteness conditions max-$\preceq$, min-$\preceq$, "having finite height" are extension closed. It is shown here that the answer is "no" for all three properties: there is a group which is a split extension of one group of height 1 by another group of height 1, and which does not satisfy max-$\preceq$ or min-$\preceq$.

1. Introduction. Previous papers [**1, 2, 3**] have been concerned with the concept of "largeness" in group theory. I will begin by reviewing the basic ideas (for further information, the reader should consult [**1**]).

A large property is a group-theoretic property $\mathcal{P}$ satisfying:

(i) if a group G has $\mathcal{P}$ and H is a group which can be mapped homomorphically onto G, then H has $\mathcal{P}$;

(ii) if K is a subgroup of finite index in a group G, then G has $\mathcal{P}$ if and only if K has $\mathcal{P}$;

(iii) if G has $\mathcal{P}$ and N is a finite subgroup of G, then G/N has $\mathcal{P}$. For a group A the *large property generated by* A (denoted $\mathcal{L}(\mathrm{A})$) is the smallest class containing A and satisfying (i)—(iii).

LEMMA. (a) *G has $\mathcal{L}(A)$ if and only if there are groups K, L, M with K of finite index in G, M a finite normal subgroup of L, L of finite index in A, L/M a homomorphic image of K.*

(b) *It may be assumed in* (a) *that either K is normal in G, or L and M are normal in A.*

For a proof see [**1**].

If G has $\mathcal{L}(A)$ we write $A \preceq G$, and we say that G is *larger than* A. If $G \preceq A$ and $A \preceq G$ then we say that G and A are *equally large*, writte $G \simeq A$. The relation

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$\preceq$ is a quasi-order on the class of groups, and $\simeq$ is the induced equivalence relation. We denote by [G] the $\simeq$ equivalence class containing G. There is then an induced partial order (also denoted $\preceq$) on the $\simeq$-equivalence classes.

For a group G, $\mathrm{Id}(G)$ denotes the *principal ideal* generated by [G]. Thus $\mathrm{Id}(G)$ consists of those $[H]$ with $[H] \preceq [G]$. We say that G satisfies max-$\preceq$ (resp. min-$\preceq$) if every properly ascending (resp. descending) chain in $\mathrm{Id}(G)$ is finite. We define the *height* of G to be the maximum of the lengths of all proper chains in $\mathrm{Id}(G)$, if this maximum exists, and ∞ otherwise. Groups of height 1 are called *atomic* (the term *minimal* is used in [**2, 3**]).

An obvious question is whether the finiteness conditions max-$\preceq$, min-$\preceq$, "having finite height", are closed under extensions. It is probably not suprising that the answer in general is "no" for all three conditions. The aim of this note is to show this in a fairly dramatic way.

THEOREM. *There is a group G_0 which is a split extension of one atomic group by another, and which does not satisfy max-$\preceq$ or min-$\preceq$.*

Remark. The structure of $\mathrm{Id}(G_0)$ will in fact be determined. Let X be a countable infinite set. Let $Y_1, Y_2 \in 2^X$. We write $Y_1 \subseteq_a Y_2$ if all but finitely many elements of Y_1 belong to Y_2, and we write $Y_1 =_a Y_2$ if $Y_1 \subseteq_a Y_2$ and $Y_2 \subseteq_a Y_1$. Then $=_a$ is a congruence on the lattice 2^X. Consider the lattice $2^X / =_a$ and form a new lattice $\mathcal{K}$ by adjoining an element 0 which is taken to be less than every other element. Then we will show that $\mathrm{Id}(G_0)$ is isomorphic to $\mathcal{K}$.

2. A Construction. Let $\pi = \{p_i : i \in I\}$ be a set of primes. Let S be a finite non-abelain simple group, let $A_i = S_{i1} \times S_{i2} \times \cdots \times S_{ip i}$ where S_{ij} is isomorphic to S, and let A be the direct product $\Pi^D_{i\in I} A_i$. Let σ_i be a p_i cycle in the symmetric group on $\{1, 2, \ldots, p_i\}$. Define G_π to be the split extension of A by the infinite cyclic group $\langle t \rangle$, where t acts on A_i, by permuting the factors according to the permutation σ_i:

$$G_\pi = \langle A, t;\ t^{-1} S_{ij} t = S_{ij\sigma_i},\ i \in I, j = 1, \ldots, p_i \rangle.$$

We note that $\langle t \rangle$ is atomic, and A atomic if it is infinite [**2**].

We obtain some information about G_π

(2.1) *The isomorphism type of G_π is independent of the particular p_i-cycles σ_i.*

Suppose $\{\tau_i : i \in I\}$ is another sollection of p_i-cycles, and let

$$G^*_\pi = \langle A, t; t^{-1} S_{ij} t = S_{ij\tau_i},\ i \in I, j = 1, \ldots, p_i \rangle.$$

For $i \in I$, let γ_i be an element of the symmetric group on $\{1, 2, \ldots, p_i\}$ such that $\gamma_i^{-1} \sigma_i \gamma_i = \tau_i$. Then the mappings

$$\psi : G_\pi \to G^*_\pi, \qquad \varphi : G^*_\pi \to G_\pi,$$

defined by:

$$\psi : S_{ij} \to S_{ij\gamma_i},\ t \mapsto t, \qquad \varphi : S_{ij} \to S_{ij\gamma_i - 1},\ t \mapsto t,$$

$(i \in I,\ j = 1, 2, \ldots, p_i)$ are mutually inverse isomorphisms.

(2.2) *The normal subgroups of* G_π *contained in* A *are precisely the subgroups* $\mathrm{sgp}\{A_j : j \in J\}$ *where* J *is a subset of* I.

It suffices to show that if N is a normal subgroup of G_π contained in A, and if the projection of N onto $A_j (j \in I)$ is non-trivial then $A_j \leq N$ Thus, supose there is an element $ha_j \in N$ where $a_j \in A_j - \{1\}$ and $h \in \mathrm{sgp}\{A_i : i \neq j\}$. Write $a_j = s_{j1} \ldots s_{jpj}$ where $s_{jl} \in S_{jl}$ for $l = 1, \ldots, p_j$. Suppose $s_{jm} \neq 1$ (at least one of the s_{jl} is non-trivial). If $s \in S_{jm}$ then $[ha_j, s] = [s_{jm}, s] \in N$. Now for some s, $[s_{jm}, s] \neq 1$ (otherwise S would have non-trivial centre) and so $S_{jm} \cap N \neq 1$. Thus $S_{jm} \leq N$ by simplicity. Hence the conjugates of S_{jm} by powers of t also lie in N. But these conjugates are $S_{j1}, \ldots, S_{jpj}$ and so $A_j \leq N$.

(2.3) *The minimal torsion subgroups of* G_π *are the groups* $A_i (i \in I)$.

Any torsion subgroup of G_π lies in A so the result follows from (2.2).

(2.4) *If* $\pi\, \pi'$ *are distinct sets of primes then* G_π *and* $G_{\pi'}$ *are not isomorphic.*

This follows from (2.3).

(2.5) *The normal subgroups of* G_π *not contained in* A *are precisely those subgroups of the form* $\mathrm{sgp}\{t^n, A_j (j \in J)\}$, *where* n *is a positive integer* J *is a subset of* I *and* $\{p_j : j \in J\}$ *contains all the primes in* π *which do not divide* n. *Such subgroups have finite index in* G.

It is easy to see that any subgroup of the form specified is normal and of finite index.

Now let N be any normal subgroup of G_π not contained in A. Then N has an element at^n with $n \geq 1$ and $a \in A$. Consider such an element where n is as small as possible. Now if $p_i \nmid n$ then $A_i \leq N$. To prove this, it suffices to show that $S_{i1} \leq N$, for then $S_{i2}, \ldots, S_{ipi} \leq N$ by conjugation by t. Let $s \in S_{i1} - \{1\}$. Then $a^{-1} s\, a \in S_{i1}$, so $t^{-n} a^{-1} s^{-1} a t^n$ is an element s' of $S_{i1} \sigma_i^n$. Since $p_i \nmid n$, σ_i^n is a p_i-cycle, so $1\sigma_i^n \neq 1$. Now $s'\, s = [at^n, s] \in N$. If $z \in S_{i1}$ then $[s'\, s, z] = [s, z] \in N$. We thus see that N contains the commuatator of any pair of elements of S_{i1}, so $S_{i1} = [S_{i1}, S_{i1}] \leq N$.

Let π_n denote the set of primes in π which do *not* divide n. Let $B_n = \mathrm{sgp}\{A_i : i \in I,\ p_i \in \pi_n\}$, $C_n = \mathrm{sgp}\{A_i : i \in I,\ p_i \notin \pi_n\}$. Then $A = B_n \times C_n$, and, by the previous paragraph, $B_n \leq N$. Write $a = bc$ with $b \in B_n,\ c \in C_n$. Since at^n and b belong to $N, ct^n \in N$. Using the fact that t^n centralizes C_n, it can be shown similarly as in (2.2), that if the projection of c onto one of the factors A_i of C_n is non-trivial, then that factor lies in N. Thus $c \in N$ and so $t^n \in N$. Now if N contains an element $a' t^m (a' \in A)$ them m must be a multiple of n, by choice of n, so $a' \in N$. Thus $N = \mathrm{sgp}\{t^n, N \cap A\}$. By (2.2) and what we have shown above, $N \cap A = \mathrm{sgp}\{A_j : j \in J\}$ where $\pi_n \subset \{p_j : j \in J\}$.

(2.6) *Let* n *be a positive integer and let* π_n, B_n, C_n *be as in* (2.5). *Then (i)* $\mathrm{sgp}\{t^n, A\} C_n \times\ \mathrm{sgp}\{t^n, B_n\}$ *and (ii)* $\mathrm{sgp}\{t^n, B_n\}$ *is isomorphic to* G_{π_n}.

(i) is obvious.

For (ii), let $\tau_i = \sigma_i^n$. Then by (2.1),

$$G_{\pi_n} = \langle B_n, x; x^{-1}S_{ij}x = S_{ij\tau_i}(i \in I,\ p_i \in \pi_n,\ j = 1, \dots, p_i)\rangle.$$

The mapping of G_{π_n} to $\mathrm{sgp}\{t^n, B_n\}$ defined by $B_n \to B_n,\ x \mapsto t^n$ is easily seen to be an isomorphism.

(2.7) *A subgroup H of finite index in G_π contains a subgroup of finite index which is normal in Γ_π, and is isomorphic to G_{π_n} for some n.*

By standard group theory, H contains a subgroup H_1 of finite index which is normal in G_π (take H_1 to be the intersection of the distinct conjugates of H). By (2.5), $\mathrm{sgp}\{t^n, B_n\} \le H_1$ for some n and by (2.6), $\mathrm{sgp}\{t^n, B_n\}$ is isomorphic to G_{π_n}.

(2.8) *An infinite quotient of G_π is isomorphic to $G_{\pi'}$ for some $\pi' \subseteq \pi$.*

Indeed, if G_π/M is an infinite quotient of G_π then by (2.2) and (2.5), $M = \mathrm{sgp}\{A_j : j \in J\}$ for some $J \subseteq I$, and so G_π/M is isomorphic to G'_π where $\pi' = \{p_i : i \in I - J\}$.

3. Proof of Theorem. We will employ the notation of §2, and we will use the results of §2, sometimes without further comment.

Let $G_0 = G_\pi$ with π the set of all primes.

First note that if H is infinite and $H \preceq G_0$, then $H \simeq G'_\pi$ for some $\pi' \subseteq \pi$. For a subgroup of finite index in G_0 contains a subgroup of finite index isomorphic to G_{π_n} for some n. Any infinite quotient of G_{π_n} is isomorphic to $G_{\pi'}$ for some $\pi' \subseteq \pi_n \subseteq \pi$. By the Lemma, the group H is equally as large as such a quotient.

Now let $\theta, \aleph$ be subsets of π. We show that $G_\aleph \preceq G_\theta$ if and only if $\aleph \subseteq_a \theta$.

Suppose that $G_\aleph \preceq G_\theta$. By the Lemma, there are groups K, L, M with K of finte index in G_θ, L of finite index in $G_\aleph$, M a finite normal subgroup of $G_\aleph$, L/M a homomorphic image of K. By (2.8) $G_\aleph/M$ is isomorphic to $G_{\aleph'}$ for some $\aleph' =_a \aleph$, so replacing $\aleph$ by $\aleph'$ we may assume that $M = 1$. Also, L contains a subgroup of finite index isomorphic to $G_{\aleph_n}$ for some n, so replacing K by the preimage of this subgroup, and replacing $\aleph$ by $\aleph_n$, we may assume that $L = G_\aleph$. Now K contains a subgroup of finite index which is normal in G_θ and isomorphic to G_{θ_m} for some m. The image T of this in $G_\aleph$ will be a normal subgroup of finite index, and so, using (2.5) and (2.6), we see that T can be mapped homomorphically onto $G_{\aleph_l}$ for some l. Thus $G_{\aleph_l}$ is a homomorphic image of G_{θ_m}. Hence $\aleph_l \subseteq \theta_m$ by (2.8) and (2.4). Consequently $\aleph \subseteq_a \theta$, as required.

Now suppose $\aleph \subseteq_a \theta$, so that $\aleph_n \subseteq \theta$ for some n. Then $G_{\aleph_n}$ is a homomorphic image of G_θ (by 2.8), and $G_{\aleph_n}$ is isomorphic to a subgroup of finite index in $G_\aleph$. Thus $G_\aleph \preceq G_\theta$.

It is now easy to show that $\mathrm{Id}(G_0)$ has the structure described in the Remark following the statement of the Theorem (The element 0 corresponds to the $\simeq$-equivalence class consisting of all finite groups).

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