PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 37 (51), 1985, pp. 51–55

FIRST ORDER CLASSES OF GROUPS HAVING NO GROUPS WITH A GIVEN PROPERTY

Nataša Božović

Abstract. A rezult of Miller [**8**], that there exists a finitely axiomatizable theory having no nontrivial models with isolvable word problem, is generalized. It is proved here that for every strong hereditary property P of fp group there exist a finitely axiomatizable first-order theory $\mathcal{I}(P)$ having no nontrivial models that enjoy P.

Investigating some problems in the theory of finitely generable groups Miller has obtained [**8**] an interesting model-theoretic result: there exists a finitely axiomatizable theory ETG* having no nontrivial models with solvable word problem. In this note we extend that result by constructing such theories for a broad class $\mathcal{S}$ of abstract properties of algebras: for each property $P \in \mathcal{S}$ a finitely axiomatizable first-order theory $\mathcal{I}(P)$ is found such that every model of $\mathcal{I}(P)$ enjoys the property P. The examples we give below are group theoretical, but the results are transferable to other algebars as well. (In particular, hereditary and residual properties, subdirect product and subdirect decomposability etc. can be defined for arbitrary algebras.)

In what follows we consider finitely generable (fg) groups only. An algebraic property P of such groups, for which it is true that whenever a group G enjoys P so does every fg subgroup of G, we call a *hereditary* property. By $\mathcal{G}$ we denote the class of all fg groups, by $\mathcal{R}$ the class of all recursively presentable (rp) groups and by $\mathcal{F}$ the class of all finitely presentable (fp) groups. Let $\mathcal{S}$ be a subclass of $\mathcal{G}$; property P is *proper* in $\mathcal{S}$ if $P \neq \mathcal{S}$ and $P \neq \{1\}$.

For a given P, the property res P is defined by

$$\mathrm{res}P(G) \Leftrightarrow \bigcap_N \{N \lhd G \mid P(G/N)\} = \{1\}.$$

Equivalently, G enjoys res P if for every nontrivial element $g \in G$ there exists a normal subroup $N_g \lhd G$ such that $g \notin N_g$ and $P(G/N_g)$. Residual properties can

---

AMS Subject Classification (1980): Primary 20 F 10, Secondary 03 C 65.

be defined in yet another way, utilizing the subdirect product (SDP) of groups. A group G is an SDP of groups $G_i(i \in I)$ if G is a subgroup of the direct product $\Pi_i G_i$ such that $\pi_i(G) = G_i$; here π_i is the i-th *projection*, $\pi_i : \Pi_i G_i \to G_i$. It can be proved (see e.g. [**3**]) that a group G enjoys res P iff G is an SDP of group possesing the property P. A group G is SDP-*decomposable* if it contains a family $\{N_i \mid i \in I\}$ of nontrivial normal subgroups N_i such that $\cap_i N_i = \{1\}$. If the opposite is true, the group is SDP-*indecomposable*; for example, such are all simple groups as well as all S_n groups with $n > 4$.

From the above definition of res P, it follows that

$$P \subseteq \text{res } P \tag{1}$$

for every $'P \subset \mathcal{G}$. In some cases, $P = \text{res}\, P$; e.g. this is true for the property "being an Abelain group" as well as for every other hereditary property closed with respect to the direct product. On the other hand, for the property 'being a finite group" evidently $P \neq \text{res}\, P$, since some residually finite groups (e.g. the free groups) are infinite.

In the sequel we investigate the following problem: if a given property P is proper in some class $\mathcal{S}$ of fg groups, whether res P is proper in $\mathcal{S}$ or not. Notice that for hereditary properties the above problem is solved for the classes $\mathcal{F}$ and $\mathcal{R}$ simultaneously: for a given hereditary property P res P is proper in $\mathcal{F}$ iff it is proper in $\mathcal{R}$. Namely, if P is hereditary it is easily seen that res P is also hereditary itself. Next, from the theorem of Higman [**6**] it follows that every group which is universal in $\mathcal{F}$ is also universal in $\mathcal{R}$. (A group G is *universal* in the class $\mathcal{S}$ if it contains, as subgroups, isomorphic copies of every group from $\mathcal{S}$.).

Lemma 1. *Let P be a hereditary property which is proper in $\mathcal{F}$ and which does not contain all rp SDP-indecomposable groups. Then res P is also proper in $\mathcal{F}$.*

Proof. Let K be an SDP-indecomposable group from $\mathcal{R}$, which does not enjoy the property P described above. Evidently, the intersection of all nontrivial normal subgroups of K cannot be trivial itself. Then the intersection of all normal subgroups of K, such that the corresponding factor-groups enjoy P, cannot be trivial either, and K does not belong to res P by definition. (In fact, if an SDP-indecomposable group G posseses a property res P it must also possess the property P itself.) Now, K can be embedded into an fp group G (see [**6**]) which does not enjoy res P either (because res P is hereditary) and hence the property res P is proper in F. □

The properties satisfying the assumptions of the above Lemma 1 will be called *strong hereditary* (SH) properties in what follows. It immediately follows from [**13**] that for every SH property P, the property res P is algorithmically unrecognizable.

Lemma 2. *Let P be an SH property. Then there exists an fp group U_P universal in $\mathcal{F}$, such that every nontrivial factor-group of U_P enjoys $\neg P$.*

Proof. Let U be a universal fp group which has a nontrivial factor group enjoying the property P. Such a group exists indeed for every P nontrivial in $\mathcal{F}$; for example, $U = U_1 \times G_1$ where U_1 is an arbitrary universal fp group, and G_1 is an fp group enjoying P. Let further $N_P = \cap_N \{N \lhd U \mid P(U/N)\}$.

Indeed, $N_P \neq \{1\}$ since U has a normal subgroup such that the corresponding factor-group enjoys P and since res P is a proper hereditary property in $\mathcal{F}$ so that it can contain no groups universal in $\mathcal{F}$. Let $v \in N_P$ and $v \neq 1$; applying the construction Ψ of Rabin [**13**] one maps a pair (U, v) effectively onto an fp group $U(v)$. Now $U < U(v)$ since $v \neq 1$ and hence $U(v)$ is also a universal fp group. We prove now that $U(v)$ has no nontrivial factor-groups with property P, so that the Lemma is proved. Assume, on the contrary, that $H' \lhd U(v)$ is a proper normal subgroup of $U(v)$ such that $P(U(v)/H')$ is true. The construction $\Psi : (U, v) \mapsto U(v)$ ensures that $U(v)/N \cong \{1\}$ if $v \in N,\ N \lhd U(v)$. Thus $v \notin H'$, because H' is a proper subgroup of $U(v)$. Let $H = U \cap H'$; then $H \lhd U$ and $H'U/H' \cong U/H$, i. e. U/H is isomorphic to a subgroup of $U(v)/H'$. This U/H is an fg group, and hence $P(U/H)$ is true as P is hereditary. Thus $N_p < H$, i. e. $v \in H$ contrary to the above assumption, implying that neither U/H nor $U(v)/H'$ can enjoy P. $\square$

In combinatorial group theory another type of universality is frequently utilized, viz. SQ-universality. A grouyp G is SQ-*universal* if every countable group can be embedded in a quotient of the group G. Now we can state the following

Corollary 1. *For every SH property P, there exists an SQ-universal fp group S_p, every factor-group of which enjoys $\neg P$.*

Proof. Let U, v be defined as above and let us apply the construction Ψ onto the pair $(U \times F_2 v)$ where F_2 is the free group with two generators. One obtains $\Psi : (U \times F_2, v) \mapsto \overline{U}$ such that $\Psi : (U \times (F_2/N), v) \mapsto \overline{U}/\overline{N}$
where $N \lhd F_2$ so that $v \notin N$ and where $\overline{N}$ is the normal closure of N in $\overline{U}$. Since F_2 is an SQ-universal group, every countable group G can be, for certain $N \lhd F_2$, embedded into F_2/N, and therefore into the group $\overline{U}/\overline{N}$ as well. Hence, $\overline{U}$ is an SQ-universal group itself, and in view of Lemma 2 every nontrivial quotient of $\overline{U}$ enjoys $\neg P$. $\square$

Notice that the group $\overline{U}$ constructed above is also a universal group; this opens an interesting question whether an SQ-universal fp group can be constructed such that it is *not* universal, and having only quotients that enjoy $\neg P$. For certain properties the answer is positive, e. q. for the property "being a finite group". Many infinite groups having no finite quotients have been described in the literature; e. q. the group $G(p, q, r)$ defined by the presentation

$$\Pi = < a, b, c; \quad a^p b = b a^{p+1},\ \ b^q c = c b^{q+1},\ \ c^r a = a r^{r+1} >$$

(see [**11**], which has a solvable word problem. Repeating the above reasoning one can prove that there exists an infinite SQ-universal group with unsolvable word problem which is not universal in $\mathcal{F}$ and which has no finite nontrivial quotients. Namely, if T is a torsion-free fp group with unsolvable word problem, then T

is not residually finite (since every residually finite fp group has a solvable word problem [**4, 9**]). The group $T_1 = T \times F_2$ is a torsion-free group also, but it is not residually finite; it has a finite quotient (since F_2 has such a quotient) and hence the intersection K of all normal subgroups of finite index in T_1 is not trivial. Let then $w \in K$, $w \neq 1$ and let $\Psi : (T_1, w) \mapsto \overline{T}$, the group $\overline{T}$ is a torsion-free SQ-universal fp group, see [**2**]. Hence, $\overline{T}$ is not a universal fp group and it has no nontrivial finite quotients. Similarly, for P "being an Abelain group" it is sufficient to choose v to belong to the commutant of the group T_1 and to construct $\Psi : (T_1, v) \mapsto T'$. For an arbitrary SH property the problem requires more extensive consideration.

Let us also remark that from Lemma 2 and Corrolary 1 it follows immediately that for every nontrivial subproperty Q of an SH property P there exists a universal fp group U_Q and an SQ-universal fp group S_Q such that neither of them has nontrivial quotients enjoying Q. The property Q need not be hereditary (it is a Markov property in $\mathcal{F}$).

Thus we conclude that there exists a universal fp group (as wellas an SQ-universal fp group) having no nontrivial solvable quotients, a universal fp group having on torsion-free quotients, or no quotients which is a one-relator group, or no rp simple quotients etc. Miller [**8**] has constructed an fp group G such that every nontrivial quotient of G has unsolvable word problem; in that example every rp quotient has the wor problem of degree $0'$. In our terminology, in [**8**] it is proved that the property R: "having solvable word problem" is an SH property. Clearly, one can construct the groups U_P and S_P for every P which is a subproperty of R; such are e.q. the properties mentioned as examples b, d—g in Corollary 2. However, the same statement is true for some other properties which are not subproperties of R; e. g. such are a and c in Corrolary 2.

Utilizing Lemma 2, in analogy with [**8**], we prove the following

THEOREM. *For every SH property P of groups there exists a finitely axiomatizable theory $\mathcal{I}(P)$ heaving no models that enjoy P.*

Proof. Let P be an SH property. In view of Lemma 2. there exists an fp group U_P together with $v \in U_P$, $v \neq 1$, such that every quotient with respect to a normal subgroup not containing v, is not trivial and enjoys $\neg P$. Let U_P be presented by

$$\Pi = < a_1, \dots, a_n;\ R_1(a_1, \dots, a_n) = 1, \dots, R_m(a_1, \dots, a_n) = 1 >$$

and let $\mathcal{I}(P)$ be a first-order theory with group axioms and

$$(\exists\, x_1, \dots, x_n)(R_1(x_1, \dots, x_n) = 1 \wedge \dots \wedge R_m(x_1, \dots, x_n) = \\ = 1 \wedge \neg(v(x_1, \dots, x_n) = 1)).$$

Let G denote an arbitrary nontrivial model of this theory (which certainly exists in view of Lemma 2). Hence G is a group in which the formula

$$(\exists\, x_1, \dots, x_n)(R_1(x_1, \dots, x_n) = 1 \wedge \dots \wedge R_m(x_1, \dots, x_n) = \\ = 1 \wedge \neg(v(x_1, \dots, x_n) = 1)).$$

is satisfiable. In other words, for some $g_1, \dots, g_n \in G$ one has $R_1(g_1, \dots, g_n) = 1, \dots, R_m(g_1, \dots, g_n) = 1$, $v(g_1, \dots, g_n) \neq 1$. Hence, there exists a subgroup H of G, generated by $g_1, \dots, g_n$ and such that $v(g_1, \dots, g_n) \neq 1$ and $v \in H$, so that H is not trivial; the relators $R_1(g_1, \dots, g_n) = 1, \dots, R_m(g_1, \dots, g_n) = 1$ (and perhaps some independent other ones, too) are true in H so that H is (isomorphic to) a quotient of the group U_P. Hence H does not enjoy the property P, and since P is hereditary the same is true for every group G containing H as *a* subgroup □

COROLLARY 2. *There exists a finitely axiomatizable theory of groups having no models which are:*

a) solvable, b) nilpotent, c) torsion-free, d) rp simple, e) one-relator, f) free product of groups with property P, $P \subseteq R$, g) direct product of groups with property $P, P \subseteq R$, where R is the property "to have a solvable word problem".

Proof. a) Being solvable is a hereditary propery. Furthermore, there exists a simple group which is not solvable (e. g. any A_n group for $n \geq 5$), and hence "being solvable" is an SH property. (Note that it is not a subproperty of R, since there exists a solvable fp group with unsolvable word problem [**5**].)

b) Nilpotent goups are solvable, and hence all residually-nilpotent groups are also residually-solvable. (That "being a nilpotent group" is an SH property can be seen in yet another way: every fg nilpotent group has a solvable word problem [**12**].)

c) Torsion-free (or locally infinite) group contains no nontrivial elements of finite order. Hence no finite SDP-indecomposable group can be torsion-free and hence this property is an SH property, too. ("Being torsion-free" is not a subproperty of R because there exist torsion-free fp groups with word problems of arbitrary degree of unsolvability [**1**].)

d)—g) These properties are subproperties of R. □

Let us mention one additional possibility of applicattion of the above theorem. Neumann [**10**] and Macintyre [**7**] have proved that for an fp group G the following conditions are equivalent:

(i) G is *a* subgroup of every existentially complete group

(i) G is an rp group with solvable word problem.

Hence, existentially complete groups are universal for the class S of all rp groups with solvable word problem. In view of the above theorem, one has the following:

COROLLARY 3. *Every nontrivial existentially complete group contains, for every SH property P, an fg subgroup enjoying the property $\neg P$.*

REFERENCES

[1] W. W. Boone, *Word problem and recursively enumerable degrees of unsolvability. A sequel on finitely presented groups*, Ann. Math. **84** (1966), 49–84.

[2] W. W. Boone, *Decision problems about algebraic and logical systems as a whole and recursively enumerable degrees of unsolvability*, Contributions to Math. Logic, (ed. by K. Schüte), North-Holland, Amsterdam, 1968.

[3] P. M. Cohn, *Universal Algebra* Harper & Row, New York, 1965.

[4] V. H. Dyson, *The word problem and residually finite groups*, Notices Amer. Math. Soc. **11** (1964), 743.

[5] O. G. Harlampovich, *Koneyecho opredelennaya razreshimaya gruppa s nerazreshimoi problemoi ravenstva*, Izv. Akad. Nauk USSR, Mat. **45** (1981), 852–873.

[6] G. Higman, *Subgroups of finitely presented groups*, Proc. Royal Soc. A, **262** (1961), 455–475.

[7] A. Macintyre, *On algebraically closed groups*, Ann. of Math. **96** (1972), 53–97.

[8] C. Miller III:"The word problem in quotient of a group", in *Aspects of effective Algebra*, Clayton, 1979, 246–250 (Upside Down, Yarra Glen, Vic. 19891).

[9] A. W. Mostowski, *On the decidability of some problems in special classes of groups*, Fund. Math. **59** (1966), 123–135.

[10] B. H. Neumann, "The isomorphism problem for algebraically closed groups", in *Word problems* (ed. by W. W. Boone, F. B. Cannonito, R. C. Lyndon) North-Holland, Amsterdam, 1973, pp. 553–562.

[11] B. H. Neumann, "On some groups presentations", in *Word Problems II*, (ed. S. I. Adyan, W. W. Boone, G. Higman), North-Holland, Amsterdam, 1980, 297–298.

[12] B. H. Neumann, *Varieties of groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 37, Springer-Verlag, Berlin, 1967.

[13] M. Rabin, *Recursive unsolvability of group theoretic problems*, Ann. Math. **67** (1958), 172–194.

Institut za matematiku
Prirodno-matematički fakultet
11000 Beograd

(Received 24 04 1984)