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ON A PROOF OF THE ERDÖS-MONK THEOREM

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Abstract. We prove an elementary proposition of combinatorial analysis, which with some use of model theory of Boolean algebras gives immediately the Erdös-Monk theorem. We shall proove also a generalization of this theorem.

Assuming the Continuum Hypothesis (CH) holds, Erdös and Monk proved [**2**] that $P(\omega)/I_0 \cong P(\omega)/I$, where $P(\omega)$ is the Boolean algebra of all subsets of ω-the set natural numbers, and I_0, I are the following ideals of $P(\omega)$:

$$I_0 = \{a \subseteq \omega : \ a \text{ is finite}\}, \qquad I = \{a \subseteq \omega : \ \sum_{n \in a} 1/n < \infty\},$$

1. An elementary statement of combinatorial analysis.

If $f, g : \ \omega \to 2, \ 2 = \{0,1\}$, then $f \leq g$ denotes $\forall n \in \omega f(n) \leq g(n)$. The following proposition may have an independent interest, so this is the reason why we exstraced it.

THEOREM 1.1. 1° *Let $f_n \in 2^\omega$, $n \in \omega$, be a sequence of functions such that*
(1) $\ldots f_2 \leq f_1 \leq f_0$, $\qquad$ (2) $\sum_{f_i(n)=1} 1/n = \infty, \ i \in \omega$.
Then there is an $h \in 2^\omega$ such that
(1′) $\sum_{h(n)=1} 1/n = \infty$, $\qquad$ (2′) $\sum_{f_i(n)<h(n)} 1/n < \infty, \ i \in \omega$.
2° *Let $f_n, \ g_n \in 2^\omega$, $n \in \omega$, be two sequences of functions such such that*
(3) $g_0 \leq g_1 \leq g_2 \leq \cdots$ *and* $\cdots \leq f_2 \leq f_1 \leq f_0$, (4) $\sum_{f_i(n)<g_i(n)} 1/n < \infty, \ i \in \omega$.
Then there is an $h \in 2^\omega$ such that
(3′) $\sum_{h(n)<g_i(n)} 1/n < \infty, \ i \in \omega$, $\qquad$ (4′) $\sum_{f_i(n)<h(n)} 1/n < \infty, \ i \in \omega$.

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Proof. 1° Let $a_i = \{n \in \omega : f_i(n) = 1\}$, $n \in \omega$. Then by the assumption on the functions f_n, we have

$$a_0 \supseteq a_1 \supseteq a_2 \supseteq \cdots \tag{5}$$

Define a sequence $b_n \subseteq \omega$ by induction in the following way. Let $b_0 \subseteq a_0$ be the (finite) subset of first elements in a_0 such that $\sum_{n \in b_0} 1/n \geq 1$. Let b_{i+1} be the subset of first elements in $a_{i+1} - (b_0 \cup \cdots \cup b_i)$ so that $\sum_{n \in b_{i+1}} 1/n \geq 1$, $i \in \omega$. The sets b_i exist by (2) and (5). Let $b = \cup_i b_i$, and define h to be characteristic function of b. Then

$$\sum_{h(n)=1} 1/n = \sum_{n \in b} 1/n = \sum_{i \in \omega} \sum_{n \in b_i} 1/n = \infty$$

i. e. (1') holds. Further, $\{n \in \omega : f_i(n) < h(n)\} \subseteq b_0 \cup \cdots \cup b_i\}$; so, the sum $\sum\limits_{f_i(n)<h(n)} 1/n$ is finite, i. e. (2') holds.

2° By (4) there exists a strictly increasing sequence $0 < s_0 < s_1 < s_2 \cdots$ of natural numbers such that

$$\sum_{\substack{f_k(n)<g_k(n) \\ s_k \leq n}} 1/n \leq 1/(k+1)^2, \quad k \in \omega.$$

Let $h \in 2^\omega$ defined by

$$h(n) = \begin{cases} 0 \text{ if } n < s_0 \\ g_k(n) \text{ iff } s_k \leq n < s_{k+1}. \end{cases}$$

Then

$$\sum_{h(n)<g_k(n)} 1/n = \sum_{\substack{h(n)<g_k(n) \\ n<s_{k+1}}} 1/n + \sum_{\substack{h(n)<g_k(n) \\ s_{k+1} \leq n}} 1/n = A + B.$$

Then A is a finite sum and $B = 0$; so (3') holds. Furthermore, let

$$\sum_{f_k(n)<h(n)} 1/n = \sum_{\substack{f_k(n)<h(n) \\ n<s_k}} 1/n + \sum_{k \leq i} \sum_{\substack{f_k(n)<h(n) \\ s_i \leq n<s_{i+1}}} 1/n = A + B.$$

Then A is a finite sum, and so $A < \infty$. Furthermore,

$$B = \sum_{k \leq i} \sum_{\substack{f_k(n)<g_i(n) \\ s_i \leq n<s_{i+1}}} 1/n \leq \sum_{k \leq i} \sum_{\substack{f_i(n)<g_i(n) \\ s_i \leq n<s_{i+1}}} 1/n \leq \sum_{k \leq i} 1/(i+1)^2 < \infty$$

i. e. (4') holds.

2. ω_1-saturated Boolean algebras

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In [**3**; Prop. 2.27] it is proved that an atomless Boolean algebra B is ω_1-saturated iff B satisfies the following condition:

H_{ω_1} (1) If $0 < \cdots < a_2 < a_1 < a_0$ is a sequence of elements of B, then there exists a $c \in B$ such that $0 < c < a_n,\ n \in \omega$.

(2) If $0 < a_0 < a_1 < \cdots \ \cdots < b_1 < b_0$ are two sequences of elements in B, then there is a $c \in B$ such that $a_n < c < b_n,\ n \in \omega$.

Using H_{ω_1} we proved in [**3**; Example 2.28] that

(1) $P(\omega)/I_0$ is an ω_1-saturated Boolean algebra.

Let D be the dual filter of i. e. $D = \{a^c : a \in I\}$. We first observe that

(2) If $f_I, g_I \in 2^\omega/I$ are such that $f_I \leq g_I$, then there is a $h \in 2^\omega$ such that $f_I = h_I$ and $h \leq g$.

To see that, let $a = \{i \in \omega : f(i) \leq g(i)\}$. Then $a \in D$, and the function h defined by $h(i) = f(i)$ if $i \in a$, and $h(i) = g(i)$ if $i \in a^c$, satisfies the required condidition.

Let $f_I,\ g_I \in P(\omega)/I$ be such that $f_I < g_I$. By (2) we may assume that $f \leq g$. Since $f_I < g_I$ we have $f_I \neq g_I$, i. e. $\{i \in \omega : f(i) = g(i)\} \notin D$, so $\{i \in \omega : f(i) \neq g(i)\} \notin I$. As $f \leq g$, then $f(i) \neq g(i)$ implies $f(i) < g(i)$, so $\sum\limits_{f(i)<g(i)} 1/n = \infty$. Thus we proved

(3) $$\text{Iff } f \leq g, \text{ then } f_I < g_I \text{ is equivalent to } \sum_{f(n)<g(n)} 1/n = \infty.$$

Finally, for $f,\ g \in 2^\omega$ we have $f_I \leq g_I$ iff $\{n : g(n) \leq f(n)\} \in D$ iff $\{n : g(n) \leq f(n)\}^c \in I$ iff $\{n : f(n) < g(n)\} \in I$ iff $\sum\limits_{f(n)<g(n)} 1/n < \infty$, i. e.

(4) $$g_I \leq f_I \text{ is equivalent to } \sum_{f(n)<g(n)} 1/n < \infty.$$

Using (2), (3), (4) and Theorem 1.1 it follows immediately that $P(\omega)/I$ satisfies the condition H_{ω_1}, therefore we have

THEOREM 2.1. *$P(\omega)/I$ is an atomless ω_1-Boolean algebra.*

If CH is assumed, then $\mid P(\omega)/I \mid = \mid P(\omega)/I_0 \mid = \omega_1$; so $P(\omega)/I_0$ and $P(\omega)/I$ are saturated Boolean algebras of the complete theory of atomless Boolean algebras; therefore by uniqueness of elementary equivalent saturated models of the given cardinality [**1**; Theorem 5.1.13] we have at once

COROLLARY 2.2. *If CH is assumed, then $P(\omega)/I_0 \cong P(\omega)/I$.*

Let us now give a generalization of the Erdös-Monk theorem. In [**4**] the notion of saturative filters is indtroduced. A filter F over a set J is k-saturative iff for every family of models A_i, $i \in J$, the reduced product $\prod_{i\in j} A_i/F$ is k-saturated. In [**4**] it is proved

THEOREM 2.3. *Assume F is a filter over a set J. Then F is k-saturative ($k > \omega$) iff F satisfies the following conditions: 1° F is k-good, 2° The reduced product $2^J/F$ is ω_1-saturated, 3° F is incomplete.*

As the proof of Lemma 4.2.2. in [**3**] shows, every filter over ω is ω_1-good. Since $I_0 \subseteq I$ by Theorem 2.1 and Theorem 2.2 we have

PROPOSITION 2.4. *The dual filter D of I is ω_1-saturative.*

COROLLARY 2.5. *Let B_i, $\in \omega$, be the Boolean algebras. Then*

1° $\prod_i B_i/D$ *is an ω_1-saturated Boolean algebra.*

2° *If CH is assumed and if for all $i \in \omega$ $\mid B_i \mid \leq \omega_1$, then $\prod_i B_i/D$ is an atomless saturated Boolean algebra of cardinality ω_1, and therefore $\prod_i B_i/D \cong 2^\omega/D(= (P\omega)/I)$ if subsets of ω are identified by their characteristic functions.*

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