

SEMANTICS FOR SOME INTERMEDIATE LOGICS

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Abstract. We give semantics for intermediate logics of the form $H + \forall S$, where $\forall S$ is the schema

$$\bigvee_{(i,j) \in S} (A_i \rightarrow A_j)$$

and S is a nonempty subset of $\{1, \dots, n\}^2$. It is proved that such a logic is complete with respect to the class of Kripke frames (X, R) which satisfy the universal closure of the formula

$$\bigvee_{(i,j),(k,i) \in S} x_{ij} R x_{ki}$$

0. Introduction

We shall give Kripke semantics for some intermediate logics, i. e., for some of the logics which lie in between the intuitionistic propositional calculus H and the classical propositional calculus. Logics for which we shall give semantics are obtained by adding to H axiom-schemata of the form

$$(\forall S) \quad \bigvee_{(i,j)} \{A_i \rightarrow A_j \mid (i,j) \in S\}$$

where S is a nonempty subset of $\{1, \dots, n\}^2$ ($n \geq 1$). Some special cases of these schemata were considered in [2, 3] and [5].

1. Preliminaries

The reader which is acquainted with Kripke semantics for H , and related techniques of canonical models and frames, can skip this section. Proofs of the results quoted here can be found in [1] and [4]. The connectives \forall, \exists and, or, \Rightarrow , *iff*, and *not* belong to the meta-logic, which is here classical. The use of the other mathematical signs should be clear from the context.

1.1. *Definition.* 1.1.1. The logic H is the propositional calculus in the language $L = \{\rightarrow, \wedge, \vee, \neg\}$ over the set of variables $V = \{p_i \mid i \in \omega\}$ given with the axiom-schemata

- (H1) $A \rightarrow (B \rightarrow A)$
(H2) $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$
(H3) $A \wedge B \rightarrow A$
(H4) $A \wedge B \rightarrow B$
(H5) $A \rightarrow (B \rightarrow A \wedge B)$
(H6) $A \rightarrow A \vee B$
(H7) $B \rightarrow A \vee B$
(H8) $(A \rightarrow C) \wedge (B \rightarrow C) \rightarrow (A \vee B \rightarrow C)$
(H9) $\neg A \rightarrow (A \rightarrow B)$
(H10) $(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)$

and the rule-schema

$$(MP) \quad \frac{A, A \rightarrow B}{B}$$

where A, B and C range over the set of all formulas, denoted by $\text{For } L$.

1.1.2. The notions of proof, proof from hypotheses and theorem are defined in the standard way. If $x \subseteq \text{For } L$ and $A \in \text{For } L$, then $x \vdash A$ stands for A is derived from x in H . The propositional calculus S is an *extension of H* iff it has the same language and variables as H , among its theorems it has all the axioms of H , and MP is its only rule. $\text{Th}(S)$ will denote the set of all theorems of S . If S is an extension of H and $x \subseteq \text{For } L$, then $x \vdash_S A$ stands for $x \cup \text{Th}(S) \vdash A$.

1.2. *Definition.* 1.2.1 $\mathfrak{X} = (X, R)$ is an H frame iff R is a reflexive and transitive relation on the nonempty set X ($\text{dom } \mathfrak{X} \stackrel{\text{def}}{=} X$, $K(H) \stackrel{\text{def}}{=} \{\mathfrak{X} \mid \mathfrak{X} \text{ is an } H \text{ frame}\}$).

1.2.2 $\mathfrak{M} = (\mathfrak{X}, v)$ is an H model iff (i) $\mathfrak{X} \in K(H)$ and (ii) $v : \text{dom } \mathfrak{X} \rightarrow P(V)$ such that

$$(\text{her}) \quad xRy \Rightarrow v(x) \subseteq v(y) \text{ for all } x, y \in \text{dom } \mathfrak{X}.$$

($\text{Fr } \mathfrak{M} \stackrel{\text{def}}{=} \mathfrak{X}$, $\text{dom } \mathfrak{M} \stackrel{\text{def}}{=} \text{dom } \text{Fr } \mathfrak{M}$, v is a valuation, $\text{val } \mathfrak{M} \stackrel{\text{def}}{=} v$, $M(H) \stackrel{\text{def}}{=} \{\mathfrak{M} \mid \mathfrak{M} \text{ is an } H \text{ model}\}$).

1.2.3. Let $\mathfrak{X} \in K(H)$ and $a \in \text{dom } \mathfrak{X}$. $\mathfrak{X}_a = (\text{dom } \mathfrak{X}_a, R_a)$ is a *truncation of the frame \mathfrak{X} at the point a* iff (i) $\text{dom } \mathfrak{X} \stackrel{\text{def}}{=} \{x \in \text{dom } \mathfrak{X} \mid aRx\}$ and (ii) $xR_a y$ iff xRy for all $x, y \in \text{dom } \mathfrak{X}_a$.

1.2.4. Let $\mathfrak{M} \in M(H)$ and $a \in \text{dom } \mathfrak{M}$. Then $\mathfrak{M}_a = ((\text{Fr } \mathfrak{M})_a, (\text{val } \mathfrak{M})_a)$ is a *truncation of the model \mathfrak{M} at the point a* iff $(\text{val } \mathfrak{M})_a(x) = \text{val } \mathfrak{M}(x)$ for all $x \in \text{dom } (\text{Fr } \mathfrak{M})_a$.

1.2.5. Let $\mathfrak{M} \in M(H)$ and $x \in \text{dom } \mathfrak{M}$, $A \in \text{For } L$. The predicate *formula* A holds in the point x of the model $\mathfrak{M}((\mathfrak{M}, x) \models A$, or, if the context is clear, briefly, $x \models A$) is defined by the recursion on the number of connectives of the formula A . The axioms of this recursion are:

$$\begin{array}{lll}
(p_i) & A = p_i, & x \models p_i \quad \text{iff} \quad p_i \in \text{val } \mathfrak{M}(x) \\
(\wedge) & A = B \wedge C, & x \models B \wedge C \quad \text{iff} \quad x \models B \text{ and } x \models C \\
(\vee) & A = B \vee C, & x \models B \vee C \quad \text{iff} \quad x \models B \text{ or } x \models C \\
(\rightarrow) & A = B \rightarrow C, & x \models B \rightarrow C \quad \text{iff} \quad \forall y(xRy \text{ and } y \models B \Rightarrow y \models C) \\
(\neg) & A = \neg B, & x \models \neg B \quad \text{iff} \quad \forall y(xRy \Rightarrow \text{not } y \models B)
\end{array}$$

(in the formulas above the quantifiers range over $\text{dom } \mathfrak{M}$)

B holds in the model $\mathfrak{M}(\mathfrak{M} \models A)$ iff $\forall x(x \in \text{dom } \mathfrak{M} \Rightarrow x \models A)$.

A holds in the frame $\mathfrak{X}(\mathfrak{X} \models A)$ iff $\forall \mathfrak{M}(\mathfrak{M} \in M(H) \text{ and } \text{Fr } \mathfrak{M} = \mathfrak{X} \Rightarrow \mathfrak{X} \models A)$.

A is H valid ($\models_H A$) iff $\forall \mathfrak{X}(\mathfrak{X} \in K(H) \Rightarrow \mathfrak{X} \models A)$.

Let $K \subseteq K(H)$; A is K valid ($K \models A$) iff $\forall \mathfrak{X}(\mathfrak{X} \in K \Rightarrow \mathfrak{X} \models A)$.

1.3. THEOREM (Intuitionistic Heredity) For all $\mathfrak{M} \in M(H)$, $A \in \text{For } L$

(Her) xRy and $x \models A \Rightarrow y \models A$ for all $x, y \in \text{dom } \mathfrak{M}$.

1.4. THEOREM $A \in \text{Th}(H) \Rightarrow \models_H A$.

1.5 THEOREM For every $\mathfrak{X} \in K(H)$, $\mathfrak{M} \in M(H)$ and $a \in \text{dom } \mathfrak{X}$ (or $\text{dom } \mathfrak{M}$).

1.5.1. $\mathfrak{X}_a \in K(H)$, 1.5.2. $\mathfrak{M}_a \in M(H)$.

1.6 THEOREM For Every $\mathfrak{M} \in M(H)$, $A \in \text{For } L$, $a \in \text{dom } \mathfrak{M}$ and every $x \in \text{dom } \mathfrak{M}$ such that aRx ,

$$(\mathfrak{M}, x) \models A \quad \text{iff} \quad (\mathfrak{M}_a, x) \models A.$$

1.7. Definition. Let S be an extension of H .

1.7.1. $\mathfrak{X}^c(S) = (X^c(S), R^c)$ is the S canonical H frame iff (i) $X^c(S)$ is the set of all S nice sets of formulas over the language L , i. e., the set of all $x \subseteq \text{For } L$ which satisfy

1° $x \vdash_S A \Rightarrow A \in x$ (x is S deductively closed) and

2° $A \vee B \in x \Rightarrow A \in x$ or $B \in x$ (x is prime) and

3° not $(\forall A \in \text{For } L)x \vdash_S A$ (x is S consistent)

(ii) $xR^c y$ iff $x \subseteq y$, for all $x, y \in X^c(S)$.

1.7.2. $\mathfrak{M}^c(S) = \mathfrak{X}^c(S), v^c$ is the S canonical model iff $v^c(x) = x \cap V$, for all $x \in X^c(S)$ (v^c is the canonical valuation).

1.8 THEOREM 1.8. *If S is an extension of H A a formula over the language L , and x a set of formulas over the language L such that not $x \vdash_S A$ then there exists an S nice set of formulas y such that $x \subseteq y$ and not $A \in y$.*

1.8.2. *If S is an extension of H , then*

$$A \in \text{Th}(S) \text{ iff } \forall x(x \text{ is } S \text{ nice} \Rightarrow A \in x).$$

1.9 THEOREM. *If S is an extension of H , then*

$$1.9.1 \ \mathfrak{X}(S) \in K(H), \quad 1.9.2 \ \mathfrak{X}^c(S) \in M(H).$$

1.10 THEOREM. *If S is an extension of H and x is an S nice set of formulas, then $(\mathfrak{M}^c(S), x) \models A$ iff $A \in x$.*

1.11 THEOREM $A \in \text{Th}(H)$ iff $\models_H A$.

1.12 THEOREM. *Let S be an extension of H and $x \subseteq \text{For } L$. Then x is S consistent iff there exists an H model \mathfrak{M} such that $\forall A(x \vdash_S A \Rightarrow \mathfrak{M} \models A)$.*

From this theorem it follows that, in particular (for $x = \emptyset$), S is H consistent iff it has an H model.

2. The logic $H + \forall S$ and its semantics

2.1 *Definition.* Let S be a nonempty subset of $\{1, \dots, n\}^2$ ($n \geq 1$). The logic $H + \forall S$ is the extension of H with the axiom-schema

$$(VS) \quad \bigvee_{(i,j)} \{A_i \rightarrow A_j \mid (i,j) \in S\}.$$

2.2 *Definition.* $Sijk$ stands for $(i,j) \in S$ and $(k,i) \in S$.

2.3 THEOREM $H + \forall S$ is consistent iff $\exists i \exists j \exists k Sijk$.

Proof. (\Rightarrow) We prove the contraposition. Suppose not $\exists i \exists j \exists k Sijk$ i.e.

$$(*) \quad \text{not } \exists i(\exists j(i,j) \in S \text{ and } \exists k(k,i) \in S).$$

Then formula $(p_0 \rightarrow p_0) \rightarrow p_0 \wedge \neg p_0$ is an instance of the schema $\forall S$ obtained by substituting $p_0 \rightarrow p_0$ for all A_i such that $\exists j(i,j) \in S$ and $p_0 \wedge \neg p_0$ for all A_i such that $\exists k(k,i) \in S$. This substitution is correct as, in the first place, such A_i 's occur as S is nonempty and only such A_i 's occur in $\forall S$; in the second, they are different because of (*). Thus, $p_0 \wedge \neg p_0$ is a theorem of $H + \forall S$; i. e. $H + \forall S$ is inconsistent.

(\Leftarrow) Suppose $\exists i \exists j \exists k Sijk$, i.e. for some (i_0, j_0, k_0) , $(i_0, j_0) \in S$ and $(k_0, i_0) \in S$. We will construct an H model for $H + \forall S$, which by Theorem 1.12, yields its consistency. This model is an element H model $(\{a\}, R, v)(xRy$ iff $x = y = a$, $v(a) = V$. As an H model, this model verifies all axioms and the rule of H . It also verifies the schema $\forall S$ as it verifies the disjunction $(A_{i_0} \rightarrow A_{j_0}) \vee (A_{k_0} \rightarrow A_{i_0})$ because it is true in all (i.e. one) points of our model. Namely, $a \models (A_{i_0} \rightarrow$

$A_{j_0} \vee (A_{k_0} \rightarrow A_{i_0})$ is equivalent with $(a \models A_{i_0} \Rightarrow a \models A_{j_0})$ or $(a \models A_{k_0} \Rightarrow a \models A_{i_0})$ which is true. This ends the proof.

2.4 Definition. Let S be a subset of $\{1, \dots, n\}^2$ ($n \geq 1$) such that $\exists i \exists j \exists k Sijk$. Then $K(\vee S)$ is the class of all H frames which satisfy the universal closure of the first-order formula

$$(k(\vee S)) \quad \bigvee_{(i,j)} \{tRx_{ij} \mid (i,j) \in S\} \Rightarrow \bigvee_{(i,j,k)} \{x_{ij}Rx_{ki} \mid Sijk\}.$$

Note that the condition imposed on S implies that the disjunction on the right-hand side of $k(\vee S)$ is nonempty. The following theorem shows that $K(\vee S)$ is the largest class of H frames which satisfy the schema $\vee S$.

2.5. THEOREM *Let S be a subset of $\{1, \dots, n\}^2$ ($n \leq 1$) such that $\exists i \exists j \exists k Sijk$. For every $\mathfrak{X} \in K(H)$*

$$\mathfrak{X} \models \vee S \text{ iff } \mathfrak{X} \in K(\vee S)$$

Proof. (\Leftarrow) Let $\mathfrak{X} \in K(\vee S)$ and let \mathfrak{M} be an H model such that $Fr\mathfrak{M} = \mathfrak{X}$. As \mathfrak{M} verifies all the axioms and the rule of H , it is enough to prove that $\mathfrak{M} \models \vee S$. Suppose the opposite. Then there exists a $t \in \text{dom } \mathfrak{M}$ such that *not* $t \models \bigvee_{(i,j) \in S} \{A_i \rightarrow A_j \mid (i,j) \in S\}$ which implies that for all $(i,j) \in S$ there exists an x_{ij} such that

$$(*) \quad tRx_{ij} \text{ and } x_{ij} \models A_i \text{ and not } x_{ij} \models A_j.$$

As $\mathfrak{X} \in K(\vee S)$ we have that $\bigwedge_{(i,j) \in S} \{tRx_{ij} \mid (i,j) \in S\}$ implies, for some i_0, j_0, k_0 such that $S_{i_0 j_0 k_0}$, $x_{i_0 j_0} Rx_{k_0 i_0}$. Because of $(*)$, $x_{i_0 j_0} \models A_{i_0}$, which, because of $x_{i_0 j_0} Rx_{k_0 i_0}$ implies, by Heredity, $x_{k_0 i_0} \models A_{i_0}$. Since this contradicts $(*)$, we have $\mathfrak{M} \models \vee S$.

(\Rightarrow) We prove the contraposition. Suppose $\mathfrak{X} \notin K(\vee S)$. Then there exist $a, b_{ij} \in \text{dom } \mathfrak{X}$ (for all $(i,j) \in S$) such that

$$(**) \quad \bigwedge_{(i,j) \in S} \{aRb_{ij} \mid (i,j) \in S\} \text{ and } \bigwedge_{(i,j,k) \in S} \{\text{not } b_{ij} Rb_{ki} \mid Sijk\}$$

Define $v : \text{dom } \mathfrak{X} \rightarrow P(V)$ with $p_i \in v(t)$ iff $\exists j((i,j) \in S \text{ and } b_{ij} Rt)$. It is easy to check that, as R transitive, v is a valuation. In the obtained model $\mathfrak{M} = (\mathfrak{X}, v)$ we have $b_{ij} \models p_i$ (for all $(i,j) \in S$, as R is reflexive) and *not* $b_{ij} \models p_j$ (for all $(i,j) \in S$, since otherwise $b_{i_0 j_0} \models p_{j_0}$, for some $(i_0, j_0) \in S$, which means $p_{j_0} \in v(b_{i_0 j_0})$, and this, by the definition of v yields $b_{j_0 k_0} Rb_{i_0 j_0}$ for some k_0 such that $(j_0, k_0) \in S$ and $S_{j_0 k_0 i_0}$, which contradicts $(**)$). Because of that we obtain aRb_{ij} and $b_{ij} \models p_i$ and *not* $b_{ij} \models p_j$, for all $(i,j) \in S$, which yields *not* $a \models p_i \rightarrow p_j$ for all $(i,j) \in S$, i.e. $\bigvee_{(i,j) \in S} \{p_i \rightarrow p_j \mid (i,j) \in S\}$ fails, in a and consequently in \mathfrak{M} , which ends the proof.

Now we prove the main theorem of this section.

2.6 THEOREM. *Let S be a subset of $\{1, \dots, n\}^2$ such that $\exists i \exists j \exists k Sijk$. For any $A \in \text{For } L$,*

$$A \in \text{Th}(H + \wedge S) \text{ iff } K(\vee S) \models A.$$

Proof. The (\Rightarrow) part follows from the previous theorem. (\Leftarrow) To prove this we prove that $\mathfrak{X}^c(H + \vee S) \in K(\vee S)$. Then we will have

$$\begin{aligned}
K(\vee S) \models A &\Rightarrow \mathfrak{X}^c(H + \vee S) \models A \\
&\Rightarrow \mathfrak{M}^c(H + \vee S) \models A \\
&\Rightarrow \forall x(x \in X^c(H + \vee S) \Rightarrow x \models A) \\
&\Rightarrow \forall x(x \in X^c(H + \vee S) \Rightarrow A \in x) \quad (\text{by 1.10}) \\
&\Rightarrow A \in Th(H + \vee S) \quad (\text{by 1.8.2})
\end{aligned}$$

So, we have to prove that $k(\vee S)$ holds in $\mathfrak{X}^c(H + \vee S)$, i. e. that

$$(*) \quad \bigwedge_{(i,j)} \{t \subseteq x_{ij} \mid (i,j) \in S\} \Rightarrow \bigvee_{(i,j,k)} \{x_{ij} \subseteq x_{ki} \mid Sijk\}$$

holds for all $H + \vee S$ nice sets $t, x_{ij} ((i,j) \in S)$. Note that R becomes \subseteq in the canonical model. Suppose that $(*)$ fails to hold for some $H + \vee S$ nice sets t, x_{ij} , i. e. that the antecedent of $(*)$ holds and that the consequent of $(*)$ doesn't hold. Then we have:

$$(1) \quad \bigwedge_{(i,j,k)} \{\text{not } x_{ij} \subseteq x_{ki} \mid Sijk\}$$

Because of (1), for any triple (i,j,k) such that $Sijk$ (at least one such triple exists because of the assumption of the theorem), there exists a formula A_{ijk} such that

$$(2) \quad A_{ijk} \in x_{ij} \text{ and not } A_{ijk} \in x_{ki}.$$

For all (i,k) such that $\exists j Sijk$ define

$$A_{ik} \stackrel{\text{def}}{=} \bigvee_j \{A_{ijk} \mid Sijk\}.$$

Obviously, $\vdash_H A_{ijk} \rightarrow A_{ik}$ for all (i,j,k) such that $Sijk$; so, as $A_{ijk} \in x_{ij}$, we have

$$(3) \quad A_{ik} \in x_{ij} \text{ for all } (i,j,k) \text{ such that } Sijk.$$

Also,

$$(4) \quad \text{not } A_{ik} \in x_{ki} \text{ for all } (i,k) \text{ such that } \exists j Sijk$$

because, in the opposite case, $A_{ik} \in x_{ki}$ implies, x_{ki} being prime (it is an $H + \vee S$ nice set), $\exists j (A_{ijk} \in x_{ki} \text{ and } Sijk)$, which contradicts (2).

For all i such that $\exists j \exists j Sijk$ define

$$A_i \stackrel{\text{def}}{=} \bigwedge_k \{A_{ik} \mid \exists j Sijk\}$$

Obviously, $\vdash A_i \rightarrow A_{ik}$ for all (i,k) such that $\exists j Sijk$; so, because of (3),

$$(5) \quad A_i \in x_{ij} \text{ for all } (i,j) \text{ such that } \exists k Sijk.$$

Also, because of (4) and the remark above,

$$(6) \quad \text{not } A_i \in x_{ki} \quad \text{for all } (i, k) \text{ such that } \exists j Sijk.$$

Formulas A_i are defined for all i such that $\exists j \exists k Sijk$, i. e. such that $\exists j(i, j) \in S$ and $\exists k(k, i) \in S$.

Define $\top \stackrel{\text{def}}{=} p_0 \rightarrow p_0$ and $\perp \stackrel{\text{def}}{=} p_0 \wedge \neg p_0$ and, for all $1 \leq i \leq n$,

$$A'_i \stackrel{\text{def}}{=} \begin{cases} A_i & \text{if } \exists j(i, j) \in S \text{ and } \exists k(k, i) \in S \in S \\ \top & \text{if } \exists j(i, j) \in S \text{ and not } \exists k(k, i) \in S \\ \perp & \text{if not } \exists j(i, j) \in S \text{ and } \exists k(k, i) \in S \\ p_0 & \text{if not } \exists j(i, j) \in S \text{ and not } \exists k(k, i) \in S. \end{cases}$$

The last case in the definition above is unimportant as such an A'_i doesn't occur in the schema $\forall S$. The formula $\bigvee_{(i,j)} \{A'_i \rightarrow A'_j \mid (i, j) \in S\}$ is a theorem of $H + \forall S$ and, consequently, belongs to all $H + \forall S$ nice sets. In particular it belongs to t and, as t is prime, for some $(i_0, j_0) \in S$, $A'_{i_0} \rightarrow A'_{j_0} \in t$. We have supposed that the antecedent of $(*)$ holds, so, $t \subseteq x_{fj}$ for all $(i, j) \in S$. In particular, $t \subseteq x_{i_0 j_0}$. Thus,

$$(7) \quad A'_{i_0} \rightarrow A'_{j_0} \in x_{i_0 j_0}.$$

Because of $(i_0, j_0) \in S$, two cases may occur: $\exists k(k, i_0) \in S$ or *not* $\exists k(k, i_0) \in S$. In the first case ($(i_0, j_0) \in S$ and $\exists k(k, i_0) \in S$) we have that $A'_{i_0} = A_{i_0}$ and $\exists k S i_0 j_0 k$. Because of (5) this implies $A'_{i_0} \in x_{i_0 j_0}$. In the second case ($(i_0, j_0) \in S$ and *not* $\exists k(k, i_0) \in S$) we have that $A'_{i_0} = \top$. Because $x_{i_0 j_0}$ contains all theorems of H , we also obtain that $A'_{i_0} \in x_{i_0 j_0}$. So, in both cases $A'_{i_0} \in x_{i_0 j_0}$ and, because of (7) we have

$$(8) \quad A'_{i_0} \in x_{i_0 j_0}.$$

Again two cases may occur: $\exists k(j_0, k) \in S$ or *not* $\exists k(j_0, k) \in S$. In the first case ($\exists k(j_0, k) \in S$ and $(i_0, j_0) \in S$) we have $A'_{i_0} = A_{i_0}$ and $\exists k S j_0 k i_0$ which brings into contradiction (6) and (8). In the second case (*not* $\exists k(j_0, k) \in S$ and $(i_0, j_0) \in S$) we have $A'_{i_0} = \perp$, which, because of (8), contradicts the consistency of $x_{i_0 j_0}$. Thus, the assumption that $(*)$ fails leads to contradiction. This ends the proof.

The previous theorem states that the logic $H + \forall S$ is complete with respect to the largest class of H frames in which $\forall S$ holds. Now we prove that this class can be made smaller and simpler.

2.7. Definition. Let S be a subset of $\{1, \dots, n\}^2$ such that $\exists i \exists j \exists k Sijk$. $K(C(\forall S))$ is the class of all H frames which satisfy the universal closure of the first-order formula

$$(C(k(\forall S))) \quad \bigvee_{(i,j,k)} \{x_{ij} R X_{ki} \mid Sijk\}$$

Note that $C(k(\mathcal{V}S))$ is the consequent of $k(\mathcal{V}S)$, so, as $C(k(\mathcal{V}S)) \Rightarrow k(\mathcal{V}S)$, we have that $K(C(\mathcal{V}S)) \subseteq K(\mathcal{V}S)$. It is easy to prove that this inclusion is proper.

2.8. THEOREM *Let S be a subset of $\{1, \dots, n\}^2$ such that $\exists i \exists j \exists k Sijk$, and let $\mathfrak{X} \in K(\mathcal{V}S)$ and $a \in \text{dom } \mathfrak{X}$. Then $\mathfrak{X}_a \in K(C(\mathcal{V}S))$.*

Proof. Because $\text{dom } \mathfrak{X}_a = \{x \in \text{dom } \mathfrak{X} \mid aRx\}$ and $\text{dom } \mathfrak{X}_a \subseteq \text{dom } \mathfrak{X}$ we have that the universal closure of $k(\mathcal{V}S)$ holds in \mathfrak{X}_a . By substituting a for t in $k(\mathcal{V}S)$ we obtain, because of the definition of \mathfrak{X}_a , that the antecedent of $k(\mathcal{V}S)$ holds for all $x_{ij} \in \text{dom } \mathfrak{X}_a$ ($i, j \in S$). Thus, in \mathfrak{X}_a holds the consequent of $k(\mathcal{V}S)$ too. This consequent is $C(k(\mathcal{V}S))$, so, $\mathfrak{X}_a \in K(C(\mathcal{V}S))$.

2.9. THEOREM. *Let S be a subset of $\{1, \dots, n\}^2$ such that $\exists i \exists j \exists k Sijk$. For any $A \in F$ or L ,*

$$A \in Th(H + \mathcal{V}S) \quad \text{iff} \quad K(C(\mathcal{V}S)) \models A.$$

Proof. We prove that $K(C(\mathcal{V}S)) \models A$ iff $K(\mathcal{V}S) \models A$, which, because of Theorem 2.6, implies our theorem. The (\Rightarrow) part follows because of $K(C(\mathcal{V}S)) \subseteq K(\mathcal{V}S)$. (\Leftarrow) Suppose *not* $K(\mathcal{V}S) \models A$. Then for some H model \mathfrak{M} such that $Fr\mathfrak{M} \in K(\mathcal{V}S)$ and for some $a \in \text{dom } \mathfrak{M}$ we have *not* $(\mathfrak{M}, a) \models A$. Because of Theorem 1.6 we obtain *not* $(\mathfrak{M}_a, a) \models A$ and, consequently *not* $Fr(\mathfrak{M}_a) \models A$. But $Fr(\mathfrak{M}_a) = (Fr(\mathfrak{M}))_a$ which by Theorem 2.8, belongs to $K(C(\mathcal{V}S))$. Thus, *not* $K(C(\mathcal{V}S)) \models A$, which ends the proof.

Let us mention, at the end of this section, that the condition $\exists i \exists j \exists k Sijk$ imposed on the index set S can be eliminated from all statements above if we define the empty disjunction as \perp and the empty conjunctions as \top . In this way the formulations (though not the proofs!) of all theorems and definitions can be made simpler. However, as this condition is, by Theorem 2.3, equivalent with the consistency of the logic $H + \mathcal{V}S$, we have decided to formulate our results for consistent logics only.

3. Some remarks

We have mentioned at the beginning of this paper that some intermediate logics of the form $H + \mathcal{V}S$ have been considered in [3] and [5]. In [5] López-Escobar has suggested that the sequence of logics $H + c_n$, obtained with

$$(c_n) \quad (A_1 \rightarrow A_2) \vee (A_2 \rightarrow A_3) \vee \dots \vee (A_{n-1} \rightarrow A_n) \vee (A_n \rightarrow A_1) \quad (n \geq 2)$$

is strictly decreasing. Boričić has proved in [3] that this is not the case. In this paper he also proves that two sequences of intermediate logics $H + a_n$ and $H + b_n$, obtained with

$$(a_n) \quad \bigvee_{1 \leq i < j \leq n} (A_i \rightarrow A_j), \quad n \geq 2$$

$$(b_n) \quad \bigvee_{1 \leq i < j \leq n} (A_i \rightarrow A_j) \vee (A_n \rightarrow A_1) \quad n \geq 2$$

and strictly decreasing and that $\bigcap_{n \geq 2} Th(H + a_n) = \bigcap_{n \geq 2} Th(H + b_n) = Th(H)$. Our paper has originated in the search for the semantics for these two sequences of logics. Now, using the results from section 2, it is easy to write down the appropriate characteristic formulas for the semantics of the logics. Let us mention that in López-Escobar's case $S = \{(1, 2), \dots, (n-1, n), (n, 1)\}$ the appropriate formula is

$$x_{i2}Rx_{n1} \vee x_{23}Rx_{12} \vee \dots \vee x_{n-1,n}Rx_{n-2,n-1} \vee x_{n1}Rx_{n-1,n}$$

which is equivalent with

$$(k_n) \quad x_1Rx_2 \vee x_2Rx_3 \vee \dots \vee x_{n-1}Rx_n \vee x_nRx_1.$$

It is easy to prove that for $n \in 2N$, k_n is equivalent with k_2 and less easy, but still elementary (the transitivity of R and invariance of k_n with respect to the cyclic permutation of the variables must be used), to prove that for $n \in 2N + 1$, k_n is equivalent with k_3 , which is another proof of the falsity of López-Escobar's suggestion. It is possible that our semantics for the intermediate logics of the form $H + \vee S$ can be used for the investigation of some classes of logics with specific S . However, no results worth mentioning are known to us.

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