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NOTE ON THE NUMBER OF SEQUENCES WITH GIVEN COMPLEXITY

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Abstract. Kolmogorov in 1964 has defined the notion of complexity of a finite binary sequence. In this paper some properties of the number of sequences with given complexity are considered.

Let S be the set of all finite binary sequences, and let $l(x)$, $x \in S$ be the length of x. Let $K(x)$ be the complexity of x with respect to some optimal function F_0^1 and let $K(x/y)$ be the conditional complexity of x given y (see [1] – [3]). It is well known that the complexity of most of the sequences is close to their length (see [2]), or

$$(\exists c)(\forall x)(K(x) \leq l(x) + c), \tag{1}$$

$$|\{x : l(x) = n,\ K(x) \geq n - m\}| \geq 2^n(1 - 1/2^m), \tag{2}$$

($|A|$ is the number of elements in A).

In [3.1.h] it was proved that

$$m - 2\log m \leq \log|\{x : K(x) \leq m\}| \leq m, \tag{3}$$

$$\log|\{x : K(x/m) \leq m\}| \asymp m. \tag{4}$$

$$(F \leq G \Leftrightarrow (\exists c)(\forall x)(F(x) \leq G(x) + c),\ F \asymp G \Leftrightarrow F \leq G \wedge G \leq F,\ \log = \log_2)$$

In this paper we consider some improvements for (3) and (4). Proofs are similar to the proofs for 1. (h) in [3], and they are based on Theorem 1.6 in [2].

(a) *Let*

$$A_n = \{x : l(x) = n,\ K(x) \geq n\}, \qquad A'_n = \{x : l(x) = n,\quad K(x/n) \geq n\},$$
$$B_n = \{x : l(x) = n,\ K(x) \geq n - 1\}, \qquad B'_n = \{x : l(x) = n,\quad K(x/n) \leq n - 1\},$$

Then

$$n - 2\log n \leq \log|A_n| \leq n, \log|A'_n| \asymp n, \tag{5}$$

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$$n-2\log n \le \log|B_n| \le n, \log|B'_n| \asymp n. \tag{6}$$

We note that this is an improvement of (2), in case $m=0$.

PROOF. We have immediately $|A_n|+|B_n|=2^n$, $\log|A_n|\le n$, $\log|B_n|\le n$, and by [3, 1. g] $\min\{K(x):l(x)=n\}\asymp K(n)\le l(n)$, which implies $(\forall n_0)(\exists n\ge n_0)(\exists x)(l(x)=n,\ K(x)<n)$, or $B_n\ne\emptyset$ for $n\ge n_0$. On the other hand $x\in B_n$ means $(\exists p_x)(l(p_x)\le n-1,\ F_0^1(p_x)=x)$ and the number of programs p_x is at the most $2^0+2^1+\ldots+2^{n-1}=2^n-1<2^n$, which implies that $0<|B_n|<2^n$, $0<|A_n|<2^n$, or $A_n\ne\emptyset$ for $n\ge n_0$.

Let, for given $p=\bar a b$, the function $F(p)$ be defined in such a way that we first choose the set A of exactly b sequences y for which $K(y)\le a$, and then $x=F(p)$ is the first y such that $y\notin A$. Then for $a=n$, $b=|B_n|$, we have $x=F(p)\notin B_n$, and $n\le K(x)\le K_F(x)\le l(p)\le\log|B_n|+2\log n$. Further, let $F'(\bar a b)=F(\bar a(2^a-b))$, and then $F'(\bar n|B_n|)=x\notin B_n$, which implies $n\le K(x)\le K_{F'}(x)\le l(\bar n|A_n|)\le \log|A_n|+2\log n$.

In the case concerning B'_n, we can take $F(b,a)=F(\bar a b)$, and then $F(|B'_n|,n)$ $=F(\bar n|B'_n|)=x\notin B'_n$, or $n\le K(x\|)\|\le K_F^-(x/n)\le l(|B'_n|)\asymp\log|B'_n|\le n$. The proof for $|A'_n|$ is similar.

(b) *Let $g(n)$ be a recursive increasing function, and $g(n)<n-1$. Let*

$$A_n=\{x:l(x)=n,\ K(x)>g(n)\},\quad A'_n=\{x:l(x)=n,\ K(x/n)>g(n)\},$$
$$B_n=\{x:l(x)=n,\ K(x)\le g(n)\},\quad B'_n=\{x:l(x)=n,\ K(x/n)\le g(n)\}.$$

Then

$$\log|A_n|\asymp\log|A'_n|\asymp n, \tag{7}$$
$$g(n)-2\log n\le\log|B_n|\le g(n),\ \ \log|B'_n|\asymp g(n). \tag{8}$$

PROOF. Let c_0 be such that $(\forall n)(\exists x)(l(x)=n,\ K(x)\le\log n+c_0)$, by [3, 1. g]. Let $g(n)-2\log n\ge c_0$. Then $B_n\ne\emptyset$, and imitating the proof in (a), we can get (8). If $g(n)-2\log n<c_0$, then (8) is trivial. Equality (7) is a direct consequence of (2), or simply $2^n\ge|A_n|=2^n-|B_n|\ge 2^n-(2^{g(n)+1}-1)\ge 2^n-2^{n-1}=2^{n-1}$, and $n\ge\log|A_n|\ge n-1$.

(c) *Let*

$$D_n=\{x:l(x)\le n,\ K(x)>n\},\qquad D'_n=\{x:l(x)\le n,\ K(x/n)>n\},$$
$$E_n=\{x:l(x)\le n,\ K(x)\le n\},\qquad E'_n=\{x:l(x)\le n,\ K(x/n)\le n\},$$
$$c_n=\max\{K(x)-n:l(x)=n\},\qquad c'_n=\max\{K(x/n)-n:l(x)=n\},$$
$$B_n=\{x:l(x)=n,\ K(x)=n+c_n\},\quad B'_n=\{x:l(x)=n,\ K(x/n)=n+c'_n\},$$
$$F_n=\{x:l(x)=n,\ K(x)>n\},\qquad F'_n=\{x:l(x)=n,\ K(x/n)>n\}.$$

Then

$$n-2\log n\le\log|B_n|\le n,\ \log|B'_n|\asymp n, \tag{9}$$

$$n - 2\log n \leq \log|D_n| \leq n,\ \log|D'_n| \asymp n, \tag{10}$$

$$n - 2\log n \leq \log|E_n| \leq n,\ \log|E'_n| \asymp n, \tag{11}$$

and for infinitely many n, $c_n > 0$, or $F_n \neq \emptyset$, and in that case

$$n - 2\log n \leq \log|F_n| \leq n,\ \log|F'_n| \asymp n. \tag{10}$$

PROOF. By definition of c_n, and by (1), $0 \leq c_n \leq c$, and $B_n \neq \emptyset$, for all n. Similarly to the previous case we can prove (9). In order to prove (10), we shall prove that $D_n \neq \emptyset$. Let $H(n) = 2^n - 1$, and $h(n) = [\sqrt{n}]$ (integer part of $\sqrt{n}$). Then $K(H(H(h(n)))) \asymp K(h(n)) \leq l(n)/2$, but $l(H(H(h(n)))) = H(h(n)) > n$, which implies that for every $n \geq n_0$ there exists an x such that $K(x) < n$ and $l(x) > n$, which is equivalent to the existence of an y such that $l(y) \leq n$ and $K(y) > n$, which means $D_n \neq \emptyset$ for $n \geq n_0$. Then, it is easy to prove (10). Furthermore, $D_n \neq \emptyset$ for $n \geq n_0$, implies that for infinitely many n we have $F_n \neq \emptyset$, and then (12).

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