A FIXED POINT THEOREM IN REFLEXIVE
BANACH SPACES

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In this note we shall prove the following fixed-point theorem.

**Theorem.** Let $B$ be a reflexive Banach space, $K$ a nonempty bounded, closed and convex subset of $B$ and let $T : K \to K$ be a map such that

$$\text{diam} \left[ T(D) \right] < \text{diam} \left( D \right)$$

(1)

holds for every closed and convex subset $D$ of $K$, containing more than one element and mapped into itself by $T$. Then $T$ has a fixed point in $K$.

**Proof.** Let $\mathcal{F}$ denote a family of all non-empty closed and convex subsets of $K$ which $T$ maps into itself. Then using the result of Smulian [7, p. 327] and Zorn’s Lemma it follows that $\mathcal{F}$ has a minimal element, say $C$. Since $T(C) \subseteq C \in \mathcal{F}$ it follows that $\text{Cl} \left[ \text{co} T(C) \right] \subseteq C$ and hence

$$T(\text{Cl} \left[ \text{co} T(C) \right]) \subseteq T(C) \subseteq \text{Cl} \left[ \text{co} T(C) \right].$$

This implies $\text{Cl} \left[ \text{co} T(C) \right] \in \mathcal{F}$ and by the minimality of $C$ we have

$$\text{Cl} \left[ \text{co} T(C) \right] = C.$$  

(2)

As $\text{diam} \left( \text{co} S \right) = \text{diam} \left( S \right)$ for every subset $S$ of $K$ [5, p. 17], (2) implies

$$\text{diam} \left[ T(C) \right] = \text{diam} \left( C \right).$$

(3)

Now, using (1) we conclude that $C$ is a singleton, say $C = u$. Therefore, $u$ is a fixed-point of $T$, and the proof is complete.

We remark that maps considered in [2], [4] and [6] satisfy the condition (1), and therefore our theorem is a certain generalization of corresponding fixed-point theorems. We shall illustrate this on a theorem given in [6].

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AMS Subject Classification (1980): Primary 47 H 10; Secondary 54 E 35; 54 H 25.
THEOREM A [6]. Let $K$ and $D$ be as in the previous theorem and $T : K \to K$ mapping satisfying the following conditions:

$$||Tx - Ty|| \leq \max\{||x - Tx||, ||y - Ty||, a||x - Ty|| + b||y - Tx||, (||x - y|| + ||x - Tx|| + ||y - Ty||)/3\};$$

$$x, y \in K, \quad a \geq 0, \quad b \geq 0, \quad a + b < 1,$$

$$\sup_{z \in D} ||z - Tz|| < r \cdot \text{diam}(D), \quad 0 < r = r(D) < 1.$$ (4.5)

Then $T$ has a unique fixed point in $K$.

**Proof.** If $\text{diam}(D) > 0$, then by (4) and (5) for every $x, y \in D$ we have

$$||Tx - Ty|| \leq \max\{r, (a + b), (1 + 2r)/3\} \cdot \text{diam}(D) < \text{diam}(D).$$

Therefore, $T$ satisfies (1), and by the previous theorem, $T$ has a fixed point in $K$. Since condition (4) implies that $T$ may have at most one fixed point, the proof is complete.

REFERENCES


