PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 36 (50), 1984, pp. 105–106

A FIXED POINT THEOREM IN REFLEXIVE BANACH SPACES

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In this note we shall prove the following fixed-point theorem.

THEOREM. *Let B be a reflexive Banach space, K a nonempty bounded, closed and convex subset of B and let $T : K \to K$ be a map such that*

$$\operatorname{diam}[T(D)] < \operatorname{diam}(D) \tag{1}$$

holds for every closed and convex subset D of K, containing more than one element and mapped into itself by T. Then T has a fixed point in K.

PROOF. Let $\mathcal{F}$ denote a family of all non-empty closed and convex subsets of K which T maps into itself. Then using the result of Smulian [7, p. 327] and Zorn's Lemma it follows that $\mathcal{F}$ has a minimal element, say C. Since $T(C) \subseteq C \in \mathcal{F}$ it follows that $\operatorname{Cl}[\operatorname{co} T(C)] \subseteq C$ and hence

$$T(\operatorname{Cl}[\operatorname{co} T(C)]) \subseteq T(C) \subseteq \operatorname{Cl}[\operatorname{co} T(C)].$$

This implies $\operatorname{Cl}[\operatorname{co} T(C)] \in \mathcal{F}$ and by the minimality of C we have

$$\operatorname{Cl}[\operatorname{co} T(C)] = C. \tag{2}$$

As $\operatorname{diam}(\operatorname{co} S) = \operatorname{diam}(S)$ for every subset S of K [5, p. 17], (2) implies

$$\operatorname{diam}[T(C)] = \operatorname{diam}(C). \tag{3}$$

Now, using (1) we conclude that C is a singleton, say $C = u$. Therefore, u is a fixed-point of T, and the proof is complete.

We remark that maps considered in [2], [4] and [6] satisfy the condition (1), and therefore our theorem is a certain generalization of corresponding fixed-point theorems. We shall illustrate this on a theorem given in [6].

AMS Subject Classification (1980): Primary 47 H 10; Secondary 54 E 35; 54 H 25.

Theorem A [6]. *Let K and D be as in the previous theorem and $T : K \to K$ mapping satisfying the following conditions:*

$$\|Tx - Ty\| \le \max\{\|x - Tx\|,\ \|y - Ty\|,\ a\|x - Ty\| + b\|y - Tx\|,$$
$$(\|x - y\| + \|x - Tx\| + \|y - Ty)/3\}; \qquad (4)$$
$$x, y \in K,\ \ a \ge 0,\ \ b \ge 0,\ \ a + b < 1,$$

$$\sup_{z \in D} \|z - Tz\| < r\ \mathrm{diam}\,(D), \quad 0 < r = r(D) < 1. \qquad (5)$$

Then T has a unique fixed point in K.

Proof. If diam $(D) > 0$, then by (4) and (5) for every $x, y \in D$ we have

$$\|Tx - Ty\| \le \max\{r, (a + b),\ (1 + 2r)/3\} \cdot \mathrm{diam}\,(D) < \mathrm{diam}\,(D).$$

Therefore, T satisfies (1), and by the previous theorem, T has a fixed point in K. Since condition (4) implies that T may have at most one fixed point, the proof is complete.

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(Received 05 10 1983)
(Revised 23 04 1984)