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A NOTE ON INVERSE-PRESERVATIONS OF REGULAR OPEN SETS

Takashi Noiri

Abstract. In this note an example is given in order to show that the following lemma is false (Kovačević [3]): If $f : X \to Y$ is an almost-continuous and almost-closed function, then $f^{-1}(V)$ is regular open (resp. regular closed) in X for every regular open (resp. regular closed) set V of Y.

1. Introduction. In 1974, the present author [5] showed that if $f : X \to Y$ is an almost-continuous and almost-open function, then $f^{-1}(V)$ is regular open (resp. regular closed) in X for each regular open (resp. regular closed) set V of Y. Recently, in [3] Kovačević has established the following lemma:

LEMMA A (Kovačević [3]). *If $f : X \to Y$ is an almost-continuous and almost-closed function, then $f^{-1}(V)$ is regular open* (*resp. regular closed*) *in X for each regular open* (*resp. regular closed*) *set V of Y.*

By making use of the preceding lemma, in [2] and [3] Kovačević has obtained the following results:

THEOREM B (Kovačević [2, 3]). *Let $f : X \to Y$ be an almost-continuous and almost-closed surjection such that $f^{-1}(y)$ is N-closed relative to X for each $y \in Y$. Then*

(1) *Y is almost-regular if so is X.*

(2) *Y is almost-regular nearly-paracompact if so is X.*

In this note, we shall give a counterexample to show that Lemma A is false, and hence the proof of Theorem B is false. The present author does not know whether Theorem B is true. However, it will be shown that Theorem B is necessarily true if the assumption *almost-continuous* is replaced by *δ-continuous.*

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2. Definitions

Throughout the present note X and Y always represent topological spaces on which no separation axioms are assumed unless explicitly stated. Let S be a subset of X. The closure (resp. the interior) of S will be denoted by $\mathrm{Cl}\,(S)$ (resp. $\mathrm{Int}\,(S)$). A subset S is said to be *regular closed* (resp. *regular open*) if $\mathrm{Cl}\,(\mathrm{Int}\,(S)) = S$ (resp. $\mathrm{Int}\,(\mathrm{Cl}\,(S)) = S$). A point $x \in X$ is said to be a *δ-cluster* point of S in X [13] if $S \cap U \neq \emptyset$ for every regular open set U containing x. If the set of all δ-cluster points of S is contained in S, then S is called *δ-closed* in X. The complement of a δ-closed set is called *δ-open*. Thus a δ-open set is the union of a family of regular open sets.

Definition 2.1. A function $f : X \to Y$ is said to be *almost-closed* (resp. *almost-open*) [12] if for every regular closed (resp. regular open) set F of X, $f(F)$ is closed (resp. open) in Y.

In [12], it was shown that every closed (resp. open) function is almost-closed (resp. almost-open) but the converses are not true in general.

Definition 2.2. A function $f : X \to Y$ is said to be *almost-continuous* [12] (resp. *δ-continuous* [8]) if for each $x \in X$ and each open neighborhood V of $f(x)$, there exists an open neighborhood V of $f(x)$, there exists an open neighborhood U of x such that $f(U) \subset \mathrm{Int}\,(\mathrm{Cl}\,(V))$ (resp. $f(\mathrm{Int}\,(\mathrm{Cl}\,(U))) \subset \mathrm{Int}\,(\mathrm{Cl}\,(V))$).

In [8] and [12], it was shown that almost-continuity is strictly weaker than each of continuity and δ-continuity which are independent of each other.

Definition 2.3. A space X is said to be *almost-regular* [10] if for each regular closed set F of X and each point $x \notin F$, there exist disjoint open sets U and V such that $F \subset U$ and $x \in V$.

Definition 2.4. A space X is said to be *nearly-paracompact* [11] if every regular open cover of X has an open locally finite refinement.

Definition 2.5. A subset K of a space X is said to be *N-closed relative to X* [1] if every cover of K by regular open sets of X has a finite subcover.

In [3], sets N-closed relative to a space are called *α-nearly compact*.

3. Results

The following example shows that Lemma A is false.

Example 3.1. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Let $f : (X, \tau) \to (X, \tau)$ be a function defined by $f(a) = f(b) = a$ and $f(c) = c$. Then f is almost-continuous and almost closed. Moreover, f is continuous, closed and δ-continuous. However, $f^{-1}(\{a\})$ is not regular open in (X, τ) although $\{a\}$ is a regular open set of (X, τ). It should be noticed that f is not almost-open.

A function $f : X \to Y$ is said to be *δ-closed* [7] if for each δ-closed set F of $X, f(F)$ is δ-closed in Y. Every δ-closed function is almost-closed but the converse is not true in general as the following example shows.

EXAMPLE 3.2. Let (X, τ) be the space in Example 3.1. Let $Y = \{x, y\}$ and $\sigma = \{\emptyset, \{x\}, Y\}$. We define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = f(b) = x$ and $f(c) = y$. Then f is almost-closed but is not δ-closed.

A function $f : X \to Y$ is called *super-continuous* [4] if for each $x \in X$ and each open neighborhood V of $f(x)$, there exists an open neighborhood U of x such that $f(\mathrm{Int}\,(\mathrm{Cl}\,(U))) \subset V$. It is known that super-continuity implies δ-continuity but the converse is not true in general [8, Example 4.4].

REMARK 3.3. Since f in Example 3.1 is super-continuous and δ-closed, we observe that Lemma A is false even when the assumption *almost-continuous* and *almost-closed* is replaced by *super-continuous* and *δ-closed*.

By using Lemma A, Kovačević has proved Theorem B and hence the proof is false. The present author does not know whether Theorem B is true. However, we can show that Theorem B is necessarily true if the assumption *almost-continuous* is replaced by *δ-continuous*.

THEOREM 3.4. *Let $f : X \to Y$ be a δ-continuous and almost-closed surjection such that $f^{-1}(y)$ is N-closed relative to X for each $y \in Y$. If X is almost-regular, then so is Y.*

Proof. Let V be a regular open set of Y and $y \in V$. Since f is δ-continuous, $f^{-1}(V)$ is δ-open in X [8, Theorem 2.2]. For each $x \in f^{-1}(y) \subset f^{-1}(V)$, there exists a regular open set W_x such that $x \in W_x \subset f^{-1}(V)$. Since X is almost-regular, by Theorem 2.2 of [10] there exists a regular open set U_x of X such that $x \in U_x \subset \mathrm{Cl}\,(U_x) \subset W_x$. The family $\{U_x \mid x \in f^{-1}(y)\}$ is a cover of $f^{-1}(y)$. Since $f^{-1}(y)$ is N-closed relative to X, there exists a finite subset K of $f^{-1}(y)$ such that $f^{-1}(y) \subset \cup\{U_x \mid x \in K\}$. Put $U = \mathrm{Int}\,(\cup\{\mathrm{Cl}\,(U_x) \mid x \in K\})$. Then U is regular open in X and $f^{-1}(y) \subset U \subset \mathrm{Cl}\,(U) \subset f^{-1}(V)$. Since f is almost-closed, $f(\mathrm{Cl}\,(U))$ is closed in Y and there exists an open set G of Y such that $y \in G$ and $f^{-1}(G) \subset U$ [5, Lemma 3]. Therefore, we have $y \in G \subset f(U) \subset f(\mathrm{Cl}\,(U)) \subset V$. Consequently, we obtain $y \in G \subset \mathrm{Cl}\,(G) \subset V$. It follows from Theorem 2.2 of [10] that Y is almost-regular.

COROLLARY 3.5. (Noiri [9]). *Let $f : X \to Y$ be a δ-continuous and δ-perfect surjection. If X is almost-regular, then so is Y.*

PROOF. A function $f : X \to Y$ is δ-perfect if and only if f is δ-closed and $f^{-1}(y)$ is N-closed relative to X for each $y \in Y$ [7, Theorem 3.5]. Since every δ-closed function, is almost-closed, this is an immediate consequence of Theorem 3.4.

THEOREM 3.6. *Let $f : X \to Y$ be a δ-continuous almost-closed surjection such that $f^{-1}(y)$ is N-closed relative to X for each $y \in Y$. If X is nearly-paracompact almost-regular, then so is Y.*

PROOF. Almost-regular of Y follows from Theorem 3.4. We shall show near-paracompactness of Y by using the fact that an almost-regular space Y is nearly-paracompact if and only if every regular open cover of Y has a locally finite refinement [11, Theorem 1.5]. Let $\mathcal{V} = \{V_\beta \mid \beta \in \omega\}$ be any regular open cover of Y. Since f is δ-continuous, $f^{-1}(\mathcal{V}) = \{f^{-1}(V_\beta) \mid \beta \in \omega\}$ is a δ-open cover of X. Since X is nearly-paracompact, by Lemma 1 of [9] $f^{-1}(\mathcal{V})$ has a regular open locally finite refinement $\mathcal{U} = \{U_\alpha \mid \alpha \in \nabla\}$. Since f is almost-closed and $f^{-1}(y)$ is N-closed relative to X for each $y \in Y$, $f(\mathcal{U}) = \{f(U_\alpha) \mid \alpha \in \nabla\}$ is a locally finite refinement of $\mathcal{V}$ [6, Lemma 2]. This shows that Y is nearly-paracompact.

COROLLARY 3.7. (Noiri [9]). *Let $f : X \in Y$ be a δ-continuous δ-perfect surjection. If X is nearly-paracompact almost-regular, then so is Y.*

PROOF. This is an immediate consequence of Theorem 3.6.

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Department of Mathematics
Yatsushiro College of Technology
Yatsushiro-shi, Kumamoto-ken
866 Japan

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