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LIFTS OF STRUCTURES ON MANIFOLDS

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Abstract. The complete and horizontal lifts of almost product, almost paracompact and para-structures on a given manifold into its tangent bundles are studied and it is shown that in most of these cases these lifts carry over the structure of M to $T(M)$. A correspondence between the integrability conditions of these structures on M and $T(M)$ is also studied.

1. Introduction. Let M be an n-dimensional differentiable manifold and let $T(M)$ be its tangent bundle. Let f be a function in M. Then the vertical lift of f, denoted by f^v in $T(M)$ is defined by $f^v = f \circ \pi : T(M) \xrightarrow{\pi} M \xrightarrow{f} R$. Let X be a vector field in M. The vertical lift of X in M to $T(M)$ denoted by X^c is defined by $X^v(iw) = (w(X))^v$, w being an arbitrary l-form in M and iw that in $T(M)$. Then we known that [5]:

$$X^v f^v = 0,\ (fX)^v = f^v X^v,\ I^v X^v = 0,\ w^v(X^v) = 0,\ (fw)^v = f^v w^v,\ [X^v, Y^v] = 0,$$
$$F^v X^v = 0,$$

where $f \in \mathcal{F}_0^0(M)$, $w \in \mathcal{F}_1^0(M)$, $F \in \mathbf{F}_1^1(M)$.

The complete lift of the function f in M to $T(M)$, denoted by f^c, is defined by $f^c = i(df)$.

The complete lift of X in M to $T(M)$, denoted by X^c, is defined by $X^c f^c = (Xf)^c$, and the complete lift of w in M to $T(M)$, denoted by w^c, is defined by $w^c(X^c) = (w(X))^c$. Then it is known that [5]:

$$(fX)^c = f^c X^v + f^v X^c, (X^c f^c) = (Xf)^c, X^c f^v = (Xf)^v,$$
$$w^v(X^c) = (w(X))^v, X^v f^c = (Xf)^v, F^v X^c = (FX)^v, F^c X^v = (FX)^v (FX)^c = F^c X^c,$$
$$w^v(X^c) = (w(X))^c, w^c(X^v) = (w(X))^v, I^c = I, I^v X^c = X^v,$$
$$[X^v, Y^v] = [X, Y]^c, [X^v, Y^c] = [X, Y]^v,$$

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Similarly, we also know that the horizontal lifts are defined by [5]

$$I^H = I, I^H X^v = X^v, I^v X^H = X^v, I^H X^H = X^H, X^H f^v = (Xf)^v,$$
$$(fX)^H = f^v X^H, w^v(X^H) = (w(X))^v,$$
$$w^H(X^v) = (w(X))^v, w^H(X^H) = 0,$$
$$F^H X^v = (FX)^v, F^H X^H = (FX)^H.$$

2. Complete and Horizontal Lifts of Almost Product Structure. If $J \in \mathcal{F}_1^1(M)$ satisfies the condition $J^2 = I$, we say that J defines an almost product structure on M. We say that J is integrable if the Nijenhuis tensor $N_j(X, Y)$ of J is identically equal to zero.

Now, we have $(J^2 - I)^c = (J^c)^2 - I$. Thus $J^2 - I = 0$ if and only if $(J^c)^2 - I = 0$ where by showing that if J is an almost product structure in M then J^c is an almost product structure in $T(M)$. Also, using the relation $(N_j)^c = N_j c$, we get that J^c in $T(M)$ is integrable if and only if J is integrable in M. Thus we have

Theorem 2.1. *For $J \in \mathcal{F}_1^1(M)$, J^c defines an almost product structure on $T(M)$, if and only if so does J on M. Moreover, J^c is integrable in $T(M)$ if and only if the same holds for J in M.*

Remark 1. Integrability of J on M implies integrability of J^c on $T(M)$ where as the converse is not true.

Remark 2. The above result is true in case of horizontal lifts too.

3. Lifts of Almost Paracontact Structures. Let an n-dimensional differentiable manifold M be endowed with a tensor field Φ of type (1, 1), a vector field ξ and a 1-form η and let them satisfy

$$\Phi^2 = I - \eta \otimes \xi, \quad \Phi(\xi) = 0, \quad \eta \circ \Phi = 0, \quad \eta(\xi) = 1. \tag{3.1}$$

Then (Φ, ξ, η) define almost paracontact structure on M.

From (3.1), we get on taking complete and vertical lifts

$$\begin{gathered}(\Phi^c)^2 = I - \eta^v \otimes \xi^c - \eta^c \otimes \xi^v \\ \Phi^c \xi^v = 0,\ \ \Phi^c \xi^c = 0,\ \ \eta^v \circ \Phi^c = 0 \\ \eta^c \circ \Phi^c = 0,\ \ \eta^v(\xi^v) = 0,\ \ \eta^v(\xi^c) = 1 \\ \eta^c(\xi^v) = 1,\ \ \eta^c(\xi^c) = 0.\end{gathered} \tag{3.2}$$

We now define a (1, 1) tensor field $\tilde{J}$ on $T(M)$ by

$$\tilde{J} = \Phi^c + \eta^v \otimes \xi^v + \eta^c \otimes \xi^c. \tag{3.3}$$

Then it is easy to show that $\tilde{J}^2X^v = X^v$ and $\tilde{J}^2X^c = X^c$, which give that $\tilde{J}$ is an almost product structure in $T(M)$.

Thus we have the following:

Theorem 3.1. *If there is an almost paracontact structure (Φ, ξ, η) in M defined by (3.1), then there exists in $T(M)$ an almost product structure defined by (3.3).*

Remark 1. From (3.3), we get

$$\begin{aligned} \tilde{J}X^v &= (\Phi X)^v + (\eta(X))^v \xi^c \\ \tilde{J}X^c &= (\Phi X)^c + (\eta(X))^v \xi^v + (\eta(X))^c \xi^c. \end{aligned} \tag{3.4}$$

From (3.1) taking horizontal lifts we get

$$\begin{aligned} (\Phi^H)^2 &= I - \eta^v \otimes \xi^H - \eta^H \otimes \xi^v \\ \Phi^H \xi^v &= 0, \;\; \Phi^H \xi^H = 0, \;\; \eta^v \circ \xi^H = 0 \\ \eta^H \circ \Phi^H &= 0, \;\; \eta^v(\xi^v) = 0, \;\; \eta^v(\xi^H) = 1 \\ \eta^H(\xi^v) &= 1, \;\; \eta^H(\xi^H) = 0 \end{aligned} \tag{3.5}$$

If we now define a (1,1) tensor field $\bar{J}$ in $T(M)$ by

$$\bar{J} = \Phi^H + \eta^v \otimes \xi^v + \eta^H \otimes \xi^H \tag{3.6}$$

then we observe that $\bar{J}^2 X^v = X^v$, $\bar{J}^2 X^H = X^H$, that is $\bar{j}$ defines an almost product structure in $T(M)$.

Calculating $\bar{J}X^v$ and $\bar{J}X^H$, we get

$$\bar{J}X^v = (\Phi X)^v + (\eta(X)\xi)^H, \quad \bar{J}X^H = (\Phi X)^H + (\eta(X)\xi)^v \tag{3.7}$$

Remark. In one of our earlier papers [4] we have defined that for every vector field A in $T(M)$

$$\pi JA = \Phi(\pi A) + \eta(KA)\xi, \quad KJA = \Phi(KA) + \eta(\pi A)\xi \tag{3.8}$$

where π and K are projection and connection map respectively as given in [1] and J is almost product structure in $T(M)$ [2]. It is interesting to note that J coincide with $\bar{J}$ defined by (3.6).

4. Complete Lift of Para-f-Structure. If M is an n-dimensional differentiable manifold endowed with a (1,1) tensor field $f \neq 0$ satisfying

$$f^3 - f = 0, \quad \operatorname{rank}(f) = r, \quad 0 < r \leq n$$

then f is called a para-f-structure of rank r and M is called a para-f-manifold.

Put $l = f^2$ and $m = I - f^2$. Then is can easily be seen that

$$l + m = I, \quad l \cdot m = m \cdot l = 0, \quad \operatorname{rank} l = r, \quad \operatorname{rank} m = n - r$$
$$l^2 = l, \quad f \cdot l = l \cdot f = f, \quad fm = mf = 0, \quad m^2 = m.$$

(4.1) show that there exist in M two complementary distributions D_l and D_m corresponding to the projection tensor l and m respectively.

When the rank of f is r, D is r-dimensional and D_m is $(n-r)$-dimensional, where dimension of $M = n$, the following integrability conditions (1), (2), (3) and (4) are known [3].

1) A necessary and sufficient condition for D_m to be integrable is that $N(mX, mY) = 0$ for any $X, Y \in \mathbf{F}_1^0(M)$ and N is Nijenhuis tensor of f i.e.

$$N(X,Y) = f^2[X,Y] + [fX, fY] - f[fX, fY] - f[X, fY]$$

2) A necessary and sufficient condition for D_l to be integrable is that $mN(X,Y) = 0$ for any $X, Y \in \mathbf{F}_0^1(M)$.

3) A necessary and sufficient condition for a para-f-structure to be partially integrable is that $N(lX, lY) = 0$ for any $X, Y \in \mathcal{F}_0^1(M)$.

4) A necessary and sufficient condition for a para-f-structure to be integrable is that $N(X,Y) = 0$ for any $X, Y \in \mathbf{F}_0^1(M)$.

We observe that $f^3 - f = 0$ and $(f^c)^3 = f^c = 0$ are equivalent. We also get that $\operatorname{rank}(f^c) = 2r$ if and only if the $\operatorname{rank}(f) = r$. Thus we have:

THEOREM 4.1. *The complete lift f^c of $f \in \mathcal{F}_1^1(M)$ is a para-f-structure on $T(M)$ if and only if f is a para-f-structure on M. Then f is of rank (r) if f^c of rank $(2r)$.*

Now let f be a para-f-structure of rank r in M. Then the complete lifts l^c of l and m^c of m are complementary projection tensors in $T(M)$. Thus there exist in $T(M)$ two complementary distributions D_{l^c} and D_{m^c} determined by l^c and m^c respectively. The distribution D_{l^c} and D_{m^c} are respectively the complete lifts of D_l^c and D_m^c of D_l and D_m. If we denote by N and $\tilde{N}$ the (4) are respectively equivalent to the following conditions:

(1′) $N^c(m^c X^c, m^c Y^c) = 0$

(2′) $m^c N^c(X^c, Y^c) = 0$

(3′) $N^c(l^c X^c, l^c Y^c) = 0$

(4′) $N^c(X^c, Y^c) = 0$ for any $X, Y \in \mathbf{F}_0^1(M)$

Then we have the following result:

PROPOSITION 4.2. *The complete lift f^c of a para-f-structure f in $T(M)$ satisfies one of the integrability conditions (1′), (2′), (3′) and (4′) if and only if f satisfies the corresponding integrability condition in M.*

5. Horizontal Lift of Para-f-Structure. If we consider the horizontal lift of $f^3 - f = 0$ we get: $(f^H)^3 - f^H = 0$ and they are found to be equivalent. Now if f has rank r, then f^H has rank $2r$. Consequently we have the following result:

THEOREM 5.1. *The horizontal lift f^H of $f \in \mathbf{F}_1^1(M)$ is a para-f-structure if and only if f is so. Then f is of rank r and only f^H of rank $2r$,*

Now, let f be a para-f-structure of rank r in M, then the horizontal lifts l^H of l and m^H of m are horizontal projections tensor in $T(M)$. Thus there exist in $T(M)$ two horizontal distributions D_{lH} and D_{mH} determinated by l^H and m^H respectively. The distributions D_{lH} and D_{mH} are respectively the horizontal lifts D_l^H and D_m^H of D_l and D_m. If we denote by N and $\bar{N}$ the Nijenhuis tensors of f and f^H respectively then conditions (1), (2), (3) and (4) respectively equivalent to the following conditions:

(1″) $N^H(m^H X^H, m^H, Y^H) = 0$

(2″) $m^H N^H(X^H, Y^H) = 0$

(3″) $N^H(l^H X^H, l^H Y^H) = 0$

(4″) $N^H(X^H, Y^H) = 0$ for $X, Y \in \mathcal{F}_0^1(M)$

Then we have the following result:

PROPOSITION 5.2. *The horizontal lift f^H of a para-f-structure f in M satisfies one of the integrability conditions (1″), (2″), (3″) and (4″) if and only if f satisfies the corresponding integrability conditions in M.*

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