PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 36 (50), 1984, pp. 79–85

ON CONVERGENCE DOMAINS OF FUNCTION METHODS

M. Lazić, L. Pevac

Abstract. We consider function convergence methods and the possibility of giving analogous formulations of Schur's Theorem for matrix convergence methods. So we give a class of function methods and conditions under which an analogue of Schur's Theorem is valid.

Let $F = (f_n(x))$ be a sequence of functions which map a set X into the complex plane C and let $\mathcal{F}$ be a filter on X.

DEFINITION 1. The complex sequence (s_n) converges to s by the function convergence method $(X, F, \mathcal{F})$ (briefly, $(s_n)F$-converges to s) if the series[1] $\sum_n f_n(x)s_n$ converges for every $x \in X$ and

$$s = \lim_{x,\mathcal{F}} \sum_n f_n(x)s_n.$$

If X is the set N of natural numbers and $\mathcal{F}$ is a Frechet filter ($G \in \mathcal{F}$ iff G is the complement of a finite subset in N), then we have the matrix convergence method associated with matrix $A = (a_{kn})$ where $a_{kn} = f_n(k)$ $(k, n \in N)$.

DEFINITION 2. The complex sequence (s_n) converges to s by matrix convergence method (briefly, (s_n) A-converges to s) if the series $\sum_n a_{kn}s_n$ converges for every $k \in N$ and

$$s = \lim_{k\to\infty} \sum_n a_{kn}s_n.$$

For the matrix convergence method defined by a matrix A Schur has proved the following result (see [3]).

THEOREM A. *Every bounded sequence is A-convergent iff the following conditions are satisfied:*

AMS Subject Classification (1980): Primary 40 C 15

[1] $\sum_n A_n$ stands for $\sum_{n=1}^{\infty} A_n$ unless stated otherwise.

1° $\lim_{n\to\infty} a_{nk} = a_k$ $(k \in N)$

2° $\sum_k |a_{nk}| < \infty$ $(n \in N)$

3° $\lim_{n\to\infty} \sum_k |a_{nk} - a_k| = 0$.

It is easy to see that the conjunction of conditions 1°, 2° and 3° is equivalent to the conjunction of 1° and

4° $\sum |a_{nk}|$ converges uniformly by n.

So the following result is valid (see (1]).

THEOREM B. *Every bounded sequence is A-convergent iff conditions 1° and 4° are satisfied.*

Let us try to formulate analogously the preceding theorems for function convergence methods. For the sake of simplicity we will consider the case when $X = (a, b](-\infty \le a < b < +\infty)$ and $\mathcal{F}$ consists the intersections of $(a, b]$ and the neighborhoods of a in the following result.

THEOREM A'. *The convergence domain $((a, b], F)^c$ of the function convergence method $((a, b], F)$ includes the set of all bounded sequences T_b iff the following conditions are satisfied:*

1′ $\lim_{x\to a+0} f_k(x) = g_k$ $(k \in N)$

2′ $\sum_k |f_k(x)| < \infty$ $(x \in X)$

3′ $\lim_{x\to a+0} \sum_k |f_k(x) - g_k| = 0$.

There is some difficulty to formulate analogously Theorem B for function convergence methods. Namely, one can observe many possibilities for adapting condition 4° such as

4′ the series

$$\sum_k |f_k(x)| \tag{$*$}$$

converges uniformly on $(a, b]$;

4″ the series $(*)$ converges on $(a, b]$ and converges uniformly on $(a, c]$ where c is fixed and $c \in (a, b]$;

4‴ the series $(*)$ converges on $(a, b]$ and converges uniformly on $\cup_p\{x_p\}$ for every sequence (x_p) on $(a, b]$ converging to a.

If the function convergence method $((a, b], F)$ is continuous in the sense of [4], then the following holds (see [2]).

THEOREM B.′ *The convergence domain of continuous function convergence method contains the set of all bounded sequences iff conditions 1′ and 4′ are satisfied.*

If a functional convergence method is not continuous, then Theorem B' is not true. Moreover we have following fact.

PROPOSITION. *There exists a function convergence method whose convergence domain contains the set of all bounded sequences but the series $(*)$ does not converge uniformly on any $(a, c]$.*

PROOF: Set

$$x_n = 2^{-n}, y_{nk} = x_{n+1} + x_{n+k+1} \qquad (n, k \in N).$$

Let us define a sequence of continuous functions of $(0, 1]$:

$$u_k(x) = d(x, (0, 1] - A_k) \qquad (k \in N)$$

where $A_k = \cup_{n=0}^{\infty}(y_{nk+1}, y_{nk})$ and d is usual distance on the real line.

The sequence $(u_k(x))$ has the following properties:

(a) $0 \leq u_k(x) \leq x$ for $x \in (0, 1]$ and $k \in N$;

(b) for every $x \in (0, 1]$ there exists a $K(x) \in N$ such that $k \neq K(x) \Rightarrow u_k(x) = 0$ for every $k \in N$;

(c) $u_k(x)$ converges pointwise to 0 on $(0, 1]$;

(d) $u_k(x)$ converges uniformly to 0 on $\cup_p\{x_p\}$, where (x_p) is an arbitrary sequence on $(0, 1]$ converging to 0;

(e) for any a where $0 < a \leq 1$, $u_k(x)$ does not converge uniformly on $(0, a]$.

Property (a) follows immediately from the definition of $u_k(x)$.

If $x \in (0, 1]$, then there exist $s, t \in N$ such that $x \in [y_{st+1}, y_{st}]$. In this case $k \neq t$ implies $u_k(x) = 0$. Hence, $K(x) = t$ and property (b) holds.

Now property (c) follows immediately from (b).

Let $x_p \to 0$ and $\varepsilon > 0$. The inequality $x_p \geq \varepsilon$ is true for at most a finite number of values of p, for instance $p_1, p_2, \dots, p_r$. Let us take

$$k(\varepsilon) = \max\{K(x_{p_1}), K(x_{p_2}), \dots, K(x_{p_r})\}$$

where $K(x_{p_i})(1 \leq i \leq r)$ are determined by property (b). Then $u_n(x_{p_i}) = 0$ for every $n > k(\varepsilon)$ and $1 \leq i \leq r$. Since we have $0 < x_p < \varepsilon$ whenever $p \neq p_i(1 \leq i \leq r)$, we have that for every $n \not\!N$ and $p \neq p_i(1 \leq i \leq r)$ the following holds:

$$\theta \leq u_n(x_p) \leq x_p < \varepsilon.$$

Finally, we conclude that for every $\varepsilon > 0$ there exists a $k(\varepsilon)$ such that $n > k(\varepsilon)$ implies $u(x_p) < \varepsilon$ for every $p \in N$. Thus (d) is proved.

To prove (e) it suffices to show that for every $0 < a \leq 1$ there exists a $\varepsilon(a) > 0$ and a sequence (z_k) from $(0, a]$ such that

$$\lim_{k\to\infty} u_k(z_k) = \varepsilon(a).$$

Indeed, for a fixed number a we choose m such that $a > 2^{-m-1}$ and set $z_k = (y_{mk+1} + y_{mk})/2$. As $z_k \to 2^{-m-1}$, all but a finite number of members of the sequence (z_k) belong to $(0, a]$. Moreover we have

$$\lim_{k\to\infty} u_k(z_k) = \lim_{k\to\infty} z_k = 2^{-m-1}.$$

Now, property (e) is proved by taking $\varepsilon(a) = 2^{-m-1}$.

The properties of the function sequence $(u_n(x))$ enable us to construct a function convergence method such that:

(i) the convergence domain contains all the bounded sequences;

(ii) the method does not satisfy condition $4''$.

Set $a = 0$, $b = 1$, $f_1(x) = u_1(x)$, $f_{k+1}(x) = u_{k+1}(x) - u_k(x)$ $(k \in N)$. The properties mentioned before imply:

(a$'$) $\lim_{x\to+0} f_k(x) = 0 \quad (k \in N)$

(b$'$) $\sum_k |f_k(x)| \le 2x \quad (x \in (0,1])$;

(c$'$) $\lim_{x\to+0} \sum_k |f_k(x) - 0| \le 2l \lim_{x\to+0} x = 0$.

Hence, the convergence domain of our method contains the set of all bounded sequences by virtue of Theorem A$'$.

Since

$$\sum_{k=0}^{n} f_k(x) = u_n(x) \quad (n \in N)$$

we conclude that the series $\sum_k f_k(x)$ does not converge uniformly on any interval $(0, a]$, $0 < a \le 1$.

Consequently, the series $\sum_k |f_k(x)|$ has the same property and this completes the proof of our proposition.

Remark 1. The previous example shows that there exists a function convergence method whose functions are continuous but the method is not continuous in the sense of [2].

Further, we can easily prove the following criterion:

Let the convergence domain of $((a,b], F)$ contain the set of all bounded sequences. If for every $\varepsilon > 0$ there exists a sequence (y_n) such that

$$0 < y_n - a < \varepsilon \ \ (n \in N) \ \text{ and } \ \liminf_{n\to\infty} f_n(y_n) = d > 0$$

then this method is not continuous.

When $f_n(x) = a_n x^n g(x)$ it is easy to check whether this method is continuous or not. If $f_n(x)$ are arbitrary functions, the checking is quite difficult. Let us consider the following class of functional convergence methods.

Definition 3. The function convergence method $((a,b], F)$ is an L-type method if the series $(*)$ converges on $(a,b]$ and if this series converges uniformly on any interval $[a_1, c]$, where $a < a_1 \le c$ and c is fixed number in $(a,b]$.

Remark 2. If there exists a decreasing sequence (a_n) from $(a,b]$ converging to a such that

$$\sum_{j=k}^{\infty} |f_j(x)| \qquad (k \in N) \tag{(**)}$$

is monotone on (a_{m+1}, a_m) $(m \in N)$, then this method is an L-type method.

According to the previous remark we can conclude that the following methods are L-type methods.

EXAMPLE 1. Let $A = (a_{ij})$ be an infinite matrix with the property $\sum_j |a_{ij}| < \infty$ $(i \in N)$. Let us define the sequence of functions $F = (f_j(x))$:

$$f_j(x) = \begin{cases} a_{ij} & \text{for } x = i \\ b_{ij}a_{ij} & \text{for } x \in (i, i+1) \end{cases} \qquad (i \in N)$$

where $|b_{ij}| < B_1$.

Condition $(**)$ is satisfied because $f_j(x)$ are constant for $x \in (i, i+1)$.

If $f_j(x)$ are continuous on $(i, i+1]$, then this method is equivalent to the matrix convergence method associated with the matrix A.

EXAMPLE 2. Let us consider the Abel method. In this case $f_j(x) = (1 - x)x^{j-1}$ $(j \in N)$ on the interval $[0, 1)$. The corresponding sum in the condition $(**)$ is equal x^{k-1}. This function is obviously monotone on $[0, 1)$.

THEOREM B$''$. *The convergence domain of an L-type method $((a, b], F)$ contains all the bounded sequences iff conditions* $1'$ *and* $4''$ *are satisfied.*

In the proof we need the following simple lemma which might be interesting by itself.

LEMMA. *Let $u_k(x)$ be a function sequence converging pointwise to $u(x)$ on $(a, b]$. Then the conditions*

(1) *$u_k(x)$ converges uniformly to $u(x)$ on $\cup_k\{a_k\}$, where $a_k \in (a, b]$ and $a_k \to a$*

(2) *$u_k(x)$ converges uniformly to $u(x)$ on each interval $[c, b]$ where $a < c \leq b$ imply the uniform convergence of the sequence $(u_k(x))$ to $u(x)$ on $(a, b]$.*

PROOF. Let us suppose that $(u_k(x))$ does not converge uniformly on $(a, b]$. Then there exists a convergent sequence (z_k) from $(a, b]$ such that

$$\lim_{k\to\infty} \inf |u_k(z_k) - u(z_k)| > 0.$$

There are two possibilities: either (z_k) converges to a, or (z_k) does not converge to a. It is easy to see that the first possibility contradicts condition (1) and the second one contradicts condition (2).

PROOF OF THE THEOREM: Let us suppose that conditions $1'$ and $4''$ are satisfied. If $(s_j) \in T_b$, then $\sum_j f_j(x)s_j$ converges for every $x \in (a, b]$. Moreover this series converges uniformly on $(a, c]$ and we have

$$\lim_{x\to a+o} \sum_j f_j(x)s_j = \sum_j g_j s_j$$

for every $(s_j) \in T_b$. So, conditions $1'$ and $4''$ are sufficient for the convergence domain of an L-type method to contain all the bounded sequences. (Obviously, $1'$ and $4''$ are sufficient for $((a, b], F)^c \supseteq T_b$ in general.)

Let us prove that conditions $1'$ and $4''$ are necessary. Therefore, suppose that every bounded sequence (s_k) is convergent by the L-type method $((a,b],F)$. From the results concerning function convergence methods in general we can see that condition $1'$ and the first part of condition $4''$ are satisfied.

Suppose that (x_i) is a sequence from $(a,b]$ converging to a. Now, we can define a matrix convergence method by

$$a_{ij} = f_j(x_i) \qquad (i,j \in N) \tag{$(***)$}$$

The convergence domain of this method contains T_b by our assumption. Applying Theorem B, we have that the series $\sum_j |f_j(x_i)|$ converges uniformly by $i \in N$. Since (x_i) is an arbitrary sequence, condition (1) of the Lemma is satisfied. Condition (2) of the Lemma is satisfied by definition of L-type method. Hence, we conclude that the second part of condition $4''$ is satisfied and this completes the proof.

Finally, for general function convergence methods we have the following theorem.

THEOREM B$'''$. *The convergence domain of a function convergence method contains all the bounded sequences iff conditions* $1'$ *and* $4''$ *are satisfied.*

PROOF OF THE SUFFICIENCY PART: If $T_b \subseteq ((a,b],F)^c$, then condition $1'$ is satisfied in virtue of Theorem A'. Let (x_p) be a sequence on $(a,b]$ converging to a. Let us define the matrix convergence method by the matrix $A = (a_{ij})$ described in $(***)$. Obviously, $A^c \supseteq ((a,b],F)^c \supseteq T_b$ and we get condition $4'''$ by applying Theorem B.

PROOF OF THE NECESSITY PART: Let (x_p) and A be the same as in the preceding part. If (s_n) is a bounded sequence, then the series $\sum_n f_n(x_p)s_n$ converges uniformly in p, because the series $\sum_n |f_n(x_p)|$ converges uniformly by p (Theorem B). Hence we have:

$$\lim_{p\to\infty} \sum_n f_n(x_p)s_n = \sum_n g_n s_n.$$

Finally, we see that the following limit exists:

$$\lim_{x\to a+0} \sum_n f_n(x)s_n.$$

REFERENCES

[1] G.H. Hardy, *Divergent Series*, Oxford, 1949.

[2] M.G. Lazić, *On some classes of linear function transformations of sequences* (I); Publ. Inst. Math. (Beograd) (N. S.) **18** (**32**) (1975), 117–124.

[3] I. Schur, *Über lineare Transformationen in der Theorie der unedlichen Reichen*, J. Reine Angew. Math. **151** (1921), 79–111.

[4] L. Wodarski, *Sur les méthodes continues de limitation* (I), Studia Math. **14** (1954), 161–187.

[5] M. Lazić, *Sur les conditions necessaires and suffisiéntes pour que le domaine d'un procédé continu contienne toutes les suites bornees*, Mat. Vesnik **7** (**22**) (**2**) (1970), 217–222.

[6] M. Lazić, *Sur les procêdês fonctionnels (de limitation)*; Mat. Vesnik **6** (**21**) (**4**) (1969), 425–436.

[7] М.Г. Лазич, *О функциональных преобразованиях последовательностей* (I); Publ. Inst. Math. (Beograd) (N.S.) **14** (**28**), (1973), 83–95.

Matematički institut
Knez Mihailova 35
Beograd

(Received 26 10 1982)
(Revised 09 05 1984)