

ASYMPTOTIC PROPERTIES OF CONVOLUTION PRODUCTS OF SEQUENCES

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Abstract. Suppose three sequences $\{a_n\}_{\mathbf{N}}$, $\{b_n\}_{\mathbf{N}}$ and $\{c_n\}_{\mathbf{N}}$ are related by the equation $c_n = \sum_{k=0}^n a_{n-k} b_k$. In this paper we examine the asymptotic behavior of c_n/a_n under various conditions on $\{a_n\}_{\mathbf{N}}$ and $\{b_n\}_{\mathbf{N}}$. If $\sum_{k=0}^{\infty} |b_k| < \infty$ we discuss conditions under which $c_n/a_n \rightarrow \sum_{k=0}^{\infty} b_k$ and give sharp rate of convergence results. From our results we obtain asymptotic expansions of the form

$$c_n = a_n \sum_{k=0}^{\infty} b_k + (a_n - a_{n-1}) \sum_{k=1}^{\infty} k b_k + O(|a_n - a_{n-1}|/n).$$

1. Introduction.

The convolution product of two sequences $\{a_n\}_{\mathbf{N}}$ and $\{b_n\}_{\mathbf{N}}$ of real numbers is the sequence $c = a * b$ defined by

$$c_n = (a * b)_n = \sum_{k=0}^n a_{n-k} b_k \quad (n \geq 0).$$

Given a sequence b one is often interested in relating the asymptotic behavior of the two sequences a and c . Results of this type have been examined by a number of authors. See for instance [1], [10], [14]. For applications in probability theory we refer to [7], [8], [11]. Also related is the paper of Pavlov [12] where the asymptotic behavior of the number of solutions to $x^k = a$ in the symmetric group of order n is considered.

In this paper we want to study the so-called regular variation of sequences.

DEFINITION. A sequence $\{r_n\}_{\mathbf{N}}$ of nonnegative real numbers varies regularly at infinity and with index $\rho \in \mathbf{R}$ if $r_n \sim n^\rho L(n)$ ($n \rightarrow \infty$) where L is slowly varying (s.v.), i.e. $\lim_{t \rightarrow \infty} L(tx)/L(t) = 1$ for all $x > 0$. We will often use the notation $r = \{r_n\} \in RV_\rho$. \square

For s.v. functions we refer to [5] and [13]. For regularly varying sequences, see [2] and [15].

The next paragraph contains our main results. Proofs are given in Section 3.

2. Main results

Let $\{a_n\}_{\mathbf{N}}$, $\{b_n\}_{\mathbf{N}}$ and $\{c_n\}_{\mathbf{N}}$ be sequences of nonnegative real numbers related by

$$c_n = (a * b)_n = \sum_{k=0}^n a_{n-k} b_k \quad (n \geq 0).$$

The next result is an immediate consequence of Feller [9, p. 447]. In it we describe the asymptotic behavior of c_n in case $\sum_{k=0}^{\infty} b_k$ may be infinite.

LEMMA 2.1. *Let $(\sum_{k=0}^n b_k)_{\mathbf{N}} \in RV_{\beta}$ with $\beta \geq 0$. Then for any $\alpha \geq 0$ and any s.v. $L(x)$ the following two statements are equivalent: as $n \rightarrow \infty$*

$$(i) \quad \sum_{k=0}^n a_k \sim n^{\alpha} L(n), \quad (ii) \quad \sum_{k=0}^n c_k \sim \frac{\Gamma(1+\alpha)\Gamma(1+\alpha)}{\Gamma(1+\alpha+\beta)} n^{\alpha} L(n) \sum_{k=0}^n b_k.$$

Furthermore, if $\{a_n\}_{\mathbf{N}}$ is nondecreasing and if $\alpha + \beta > 0$, then (i) is equivalent to

$$(iii) \quad c_n \sim (\alpha + \beta) \frac{\Gamma(1+\alpha)\Gamma(1+\beta)}{\Gamma(1+\alpha+\beta)} n^{\alpha-1} L(n) \sum_{k=0}^n b_k \quad (n \rightarrow \infty). \quad \square$$

If $\{a_n\}_{\mathbf{N}}$ is nonincreasing, the following analogue of lemma 2.1 (i) \Leftrightarrow (iii) will be proved.

THEOREM 2.2. *$\{b_n\}_{\mathbf{N}} \in RV_{\beta}$ with $\beta > -1$ and let $\{a_n\}_{\mathbf{N}}$ be nonincreasing. For any α , $0 \leq \alpha \leq 1$ and any s.v. $L(x)$ the following two statements are equivalent: as $n \rightarrow \infty$*

$$(i) \quad \sum_{k=0}^n a_k \sim n^{\alpha} L(n) \quad (ii) \quad c_n \sim \frac{\Gamma(1+\alpha)\Gamma(1+\beta)}{\Gamma(1+\alpha+\beta)} n^{\alpha} L(n) b_n. \quad \square$$

In case $\sum_{k=0}^{\infty} b_k < \infty$ (i.e. $\beta = 0$) and $\{a_n\}_{\mathbf{N}} \in RV_{\alpha}$, $\alpha > 0$, lemma 2.1 (iii) gives

$$\lim_{n \rightarrow \infty} c_n / a_n = \sum_{k=0}^{\infty} b_k.$$

The same result also holds in case $\{a_n\}_{\mathbf{N}} \in RV_{\alpha}$, $\alpha \leq 0$, if we assume $\sum_{k=0}^{\infty} k^{-\alpha+\varepsilon} b_k < \infty$ for some $\varepsilon > 0$.

LEMMA 2.3. $\{a_n\}_{\mathbf{N}} \in RV_\alpha$, $\alpha \in \mathbf{R}$ and let $\{b_n\}_{\mathbf{N}}$ be a sequence of real numbers such that $\sum_{k=0}^{\infty} |b_k| < \infty$. In case $\alpha < 0$ assume that $\sum_{k=0}^{\infty} k^{-\alpha+\varepsilon} |b_k| < \infty$ for some $\varepsilon > 0$. Then

$$\lim_{n \rightarrow \infty} c_n/a_n = \sum_{k=0}^{\infty} b_k. \quad \square$$

The rate at which c_n/a_n converges to $\sum_{k=0}^{\infty} b_k$ heavily depends on the rate at which a_{n+1}/a_n converges to 1. More precisely we shall prove

THEOREM 2.4. Suppose $\{a_n\}_{\mathbf{N}}$ is a sequence of positive real numbers such that for some $\beta \in \mathbf{R}$,

$$\lim_{n \rightarrow \infty} n(a_{n-1}/a_n - 1) = \beta. \quad (2.1)$$

Let $\{b_n\}$ be a sequence of real numbers such that $\sum_{k=1}^{\infty} k|b_k| < \infty$ and in case $\beta \leq 0$ such that $\sum_{k=1}^{\infty} k^{\beta+1+\varepsilon} |b_k| < \infty$ for some $\varepsilon > 0$. Then

$$\lim_{n \rightarrow \infty} n \left(\frac{c_n}{a_n} - \sum_{k=0}^{\infty} b_k \right) = \beta \sum_{k=1}^{\infty} kb_k. \quad \square \quad (2.2)$$

This result should be compared with Theorem 2 of Bojanić and Lee [1]. There it is assumed that for $n \geq 1$, $|a_{n-1}/a_n - 1| \leq \delta_{n-1}$, where $\{\delta_n\}_{\mathbf{N}}$ is some sequence of positive numbers such that as $n \rightarrow \infty$, $n\delta_n = O(1)$ and $n(\delta_{n-1}/\delta_n - 1) = O(1)$. Under these conditions and the condition $\sum_{k=1}^{\infty} k^\alpha |b_k| < \infty$ for all $\alpha > 0$, Bojanić and Lee prove that as $n \rightarrow \infty$,

$$\left| \frac{c_n}{a_n} - \sum_{k=0}^{\infty} b_k \right| = O(\delta_n).$$

Using (2.1) we get the exact asymptotic result (2.2). Note that (2.1) implies that $\{a_n\}_{\mathbf{N}} \in RV_{-\beta}$ and in case $\beta \neq 0$ (2.1) is equivalent to the regular variation of $\{|a_{n-1} - a_n|\}_{\mathbf{N}}$.

Again talking differences of the sequence $\{a_{n-1} - a_n\}_{\mathbf{N}}$ we can obtain rate of convergence results for (2.2) also. More generally, for $n \geq 0$ let $p_{0,n} = a_n$ and $r_{0,n} = b_n$. For $k \geq 0$ define sequences $p_k = \{p_{k,n}\}_{\mathbf{N}}$ and $r_k = \{r_{k,n}\}_{\mathbf{N}}$ by induction as follows

$$p_{k+1,n} = \begin{cases} p_{k,n-1} - p_{k,n} & \text{if } n \geq k+1 \\ 0 & \text{elsewhere} \end{cases}$$

$$r_{k+1,n} = \sum_{p=n+1}^{\infty} r_{k,p} \quad n \geq 0.$$

An application of Theorem 2.4 then yields

COROLLARY 2.5. Suppose for some $k \geq 1$ $\{p_{k,n}\}_{\mathbf{N}}$ is a sequence of nonnegative real numbers such that for some $\beta \in \mathbf{R}$,

$$\lim_{n \rightarrow \infty} n \frac{p_{k,n-1} - p_{k,n}}{p_{k,n}} = \beta. \quad (2.3)$$

Let $\{b_n\}_{\mathbf{N}}$ be a sequence of real numbers such that $\sum_{n=1}^{\infty} n^{k+1}|b_n| < \infty$ and in case $\beta \geq 0$ such that $\sum_{n=1}^{\infty} n^{k+1+\beta+\varepsilon}|b_n| < \infty$ for some $\varepsilon > 0$. Then

$$\lim_{n \rightarrow \infty} n \left(\left(c_n - \sum_{m=0}^{k-1} p_{m,n} R_m(1) \right) / p_{k,n} - R_k(1) \right) = \beta R_{k+1}(1) \quad (2.4)$$

where $R_m(1) = \sum_{n=0}^{\infty} r_{m,n}$ ($m = 0, 1, \dots, k+1$). \square

In the case $k = 1$, Corollary 2.5 gives

$$\lim_{n \rightarrow \infty} n \left(\left(c_n - a_n \sum_{k=0}^{\infty} b_k \right) / (a_{n-1} - a_n) - \sum_{k=1}^{\infty} k b_k \right) = \beta \sum_{k=2}^{\infty} \frac{k(k-1)}{2} b_k.$$

Of course a similar corollary can be stated in terms of the sequence d_k defined by $d_{0,n} = a_n$ and for $k \geq 0$,

$$d_{k+1,n} = \begin{cases} d_{k,n-1} - d_{k,n} & (n \geq k+1) \\ 0 & (n < k+1). \end{cases}$$

In this case the result is

$$\lim_{n \rightarrow \infty} \left((-1)^k \left(c_n - \sum_{m=0}^{k-1} (-1)^m d_{m,n} R_m(1) \right) / d_{k,n} - R_k(1) \right) = \beta R_{k+1}(1). \quad (2.5)$$

For another result of this type, see Theorem 2.8 below.

The main disadvantage of Theorem 2.4 and its corollaries is that it does not include sequences of the form $a_n = n^\beta e^{-n^\alpha}$ ($0 < \alpha < 1, \beta \in \mathbf{R}$). In Theorem 2.6 below we shall formulate a general result that deals with sequences which satisfy one or more of the properties listed below. Let us assume that $a_n \neq 0$ for all $n \geq N$ and let us consider the following properties

$$(P1) \quad \sup_{n \geq N} |a_{n-1}/a_n| = m < \infty$$

$$(P2) \quad \sup_{n \geq N} \left(\sum_{k=0}^n |a_{n-k}| |a_k| \right) / |a_n| = M < \infty$$

$$(P3) \quad \lim_{n \rightarrow \infty} a_{n-1}/a_n = 1$$

$$(P4) \quad \lim_{n \rightarrow \infty} \left(\sum_{k=0}^n a_{n-k} a_k \right) / a_n = \lim_{n \rightarrow \infty} \frac{(a * a)}{a_n} = 2 \sum_{k=0}^{\infty} a_k, \quad \text{with } \sum_0^{\infty} |a_k| < \infty.$$

In case of sequences of positive numbers, sequences that satisfy (P3) and (P4) have been studied already. We refer to the basic paper [3] and also to [4], [6], [8], where some applications in probability theory are given. Now we prove

THEOREM 2.6. $\{a_n\}_{\mathbf{N}}$ and $\{b_n\}_{\mathbf{N}}$ be sequences of real numbers such that $a_n \neq 0$ for $n \geq N$, $\sum_{k=0}^{\infty} |a_k| < \infty$ and $\sum_{k=0}^{\infty} |b_k| < \infty$.

(i) If $\{a_n\}_{\mathbf{N}}$ satisfies (P1), (P2) and if $\sup_{n \geq N} |b_n|/|a_n| < \infty$, then also

$$\sup_{n \geq N} |c_n|/|a_n| < \infty.$$

(ii) If $\{a_n\}_{\mathbf{N}}$ satisfies (P2), (P3) and if $\lim_{n \rightarrow \infty} b_n/a_n = 0$, then

$$\lim_{n \rightarrow \infty} c_n/a_n = \sum_{k=0}^{\infty} b_k.$$

(iii) If $\{a_n\}_{\mathbf{N}}$ satisfies (P2), (P3) and (P4), and if $\lim_{n \rightarrow \infty} b_n/a_n = \alpha$ for some $\alpha \in \mathbf{R}$, then

$$\lim_{n \rightarrow \infty} c_n/a_n = \sum_{k=0}^{\infty} b_k + \alpha \sum_{k=0}^{\infty} a_k. \quad \square$$

In the special case of regularly varying sequences we obtain the following extension of Lemma 2.3.

COROLLARY 2.7. Let $\{a_n\}_{\mathbf{N}}$ and $\{b_n\}_{\mathbf{N}}$ be sequences of nonnegative real numbers such that $\sum_{n=0}^{\infty} a_n < \infty$ and $\sum_{n=0}^{\infty} b_n < \infty$. If for some $\beta \leq -1$, $\{a_n\}_{\mathbf{N}} \in RV_{\beta}$ and if $b_n/a_n \rightarrow \alpha$ ($n \rightarrow \infty$) for some $\alpha \geq 0$, then also $\{c_n\}_{\mathbf{N}} \in RV_{\beta}$ and $c_n/a_n \rightarrow \sum_{k=0}^{\infty} b_k + \alpha \sum_{k=0}^{\infty} a_k$ ($n \rightarrow \infty$). \square

Corollary 2.7 follows at once from Theorem 2.6 (iii) since regularly varying sequences obviously satisfy (P3) and (P4). Further examples of sequences satisfying (P3) and (P4) are (see e.g. [3]):

- (i) $a_n = n^{\beta} e^{-n^{\alpha}}$, $0 < \alpha < 1$, $\beta \in \mathbf{R}$
- (ii) $a_n = \exp -n(\log n)^{-\beta}$, $\beta > 0$
- (iii) sequences $\{a_n\}_{\mathbf{N}}$ such that $a_{n+1} \sim a_n$ ($n \rightarrow \infty$) and

$$\sup_{1 \leq k \leq [n/2]} a_{n-k}/a_n \leq K < \infty.$$

Using Theorem 2.6 we can also generalise Theorem 2.4 and its corollaries. With p_k and r_k as before we have

COROLLARY 2.8. Suppose for some $k \geq 1$ that $\{p_{k,n}\}_{\mathbf{N}}$ and $\{r_{k,n}\}_{\mathbf{N}}$ are such that $\sum_{n=0}^{\infty} |p_{k,n}| < \infty$ and $\sum_{n=0}^{\infty} |r_{k,n}| < \infty$.

(i) If $\{p_{k,n}\}_{\mathbf{N}}$ satisfies (P1), (P2) and if $\sup_{n \geq N} |r_{k,n}/p_{k,n}| < \infty$, then

$$\sup_{n \geq N} \left| \left(c_n - \sum_{m=0}^{k-1} p_{m,n} R_m(1) \right) / p_{k,n} \right| < \infty.$$

(ii) If $\{p_{k,n}\}_{\mathbf{N}}$ satisfies (P2), (P3), (P4) and if $r_{k,n}/p_{k,n} \rightarrow \alpha$ ($n \rightarrow \infty$), then

$$\lim_{n \rightarrow \infty} \left(c_n - \sum_{m=0}^{k-1} p_{m,n} R_m(1) \right) / p_{k,n} = R_k(1). \quad \square$$

When $k = 1$, Corollary 2.8 (i) gives

$$\sup_{n \geq N} \left| \left(c_n - a_n \sum_{k=0}^{\infty} b_k \right) / (a_{n-1} - a_n) \right| < \infty.$$

Hence, if $\sup_{n \geq N} \delta_n^{-1} |(a_{n-1} - a_n)/a_n| < \infty$ we obtain Bojanić and Lee's result mentioned before. Corollary 2.8 (ii) in case $k = 1$, gives

$$\lim_{n \rightarrow \infty} \left(c_n - a_n \sum_{k=0}^{\infty} b_k \right) / (a_{n-1} - a_n) = \sum_{k=0}^{\infty} k b_k$$

which in case (2.1) holds, gives (2.2).

Before proving these results it is worthwhile noting that our theorems can be extended as follows. Let $\{a_n\}_{\mathbf{N}}$ satisfy the following properties

$$(P'1) \quad \sup_{n \geq N} \left(\sum_{k=0}^n |a_{n-k}| |a_k| \right) / |a_n| = M < \infty$$

$$(P'3) \quad \lim_{n \rightarrow \infty} a_{n+1}/a_n = 1/r (> 0), \quad \sum_{n=0}^{\infty} |a_n| r^n < \infty$$

$$(P'4) \quad \lim_{n \rightarrow \infty} (a * a)_n / a_n = 2 \sum_{k=0}^{\infty} a_k r^k =: 2A(r).$$

Upon transforming a sequence $\{d_n\}_{\mathbf{N}}$ to the sequence $\{d'_n\}_{\mathbf{N}}$ defined by $d'_n = r^n d_n$ we obtain the following result as an immediate consequence of Theorem 2.6.

THEOREM 2.9. *Let $\{b_n\}_{\mathbf{N}}$ and $\{a_n\}_{\mathbf{N}}$ be sequences of real numbers and suppose $\{a_n\}_{\mathbf{N}}$ satisfies (P'1), (P'3), (P'4). If $\lim_{n \rightarrow \infty} b_n/a_n = \alpha$ for some $\alpha \in \mathbf{R}$, then*

$$\lim_{n \rightarrow \infty} c_n/a_n = B(r) + \alpha A(r)$$

where $B(r) = \sum_0^{\infty} b_n r^n$ and $A(r) = \sum_0^{\infty} a_n r^n$. \square

3. Proofs

3.1. Proof of Theorem 2.2. (ii) \Rightarrow (i) This follows immediately from lemma 2.1. (i) \Rightarrow (ii) The proof is divided into two parts.

Part 1: $\alpha = 0$. First observe that (i) implies

$$a_n \leq 2n^{-1} L(n) \tag{3.1}$$

for n large enough. For any fixed $\varepsilon, t, 0 < \varepsilon < t < 1$ write

$$c_n = \left(\sum_{k=0}^{[n\varepsilon]} + \sum_{k=[n\varepsilon]+1}^{[nt]} + \sum_{k=[nt]+1}^{[n]} \right) a_{n-k} b_k =: I_1 + I_2 + I_3.$$

First consider I_1 . From the monotonicity of $\{a_n\}_{\mathbf{N}}$ and (3.1) we have

$$0 \leq I_1 \leq a_{n-[n\varepsilon]} \sum_{k=0}^{[n\varepsilon]} b_k \leq 2(n-[n\varepsilon])^{-1} L(n-[n\varepsilon]) \sum_{k=0}^{[n\varepsilon]} b_k.$$

Since $\{b_n\}_{\mathbf{N}} \in RV_\beta$ and L is s.v. we obtain

$$0 \leq \limsup_{n \rightarrow \infty} \frac{I_1}{L(n)b_n} \leq \frac{2\varepsilon^{\beta+1}}{(1-\varepsilon)(1+\beta)}. \quad (3.2)$$

To handle I_2 and I_3 we use the fact that nb_n asymptotically equals a nondecreasing sequence $B(n)$. This follows from the representation theorem for regularly varying sequences [2] and $1+\beta > 0$. Hence for any $\delta > 0$ and n large we have

$$(1-\delta)B(n) \leq nb_n \leq (1+\delta)B(n).$$

Using this and the monotonicity of $B(n)$ we obtain

$$(1+\delta)B([nt]+1)n^{-1} \sum_{k=0}^{n-[nt]-1} a_k \leq I_3 \leq (1+\delta)B(n)([nt]+1)^{-1} \sum_{k=0}^{n-[nt]-1} ak. \quad (3.3)$$

Hence

$$(1-\delta)t^{\beta+1} \leq \lim_{n \rightarrow \infty} \left[\sup \right] \frac{I_3}{L(n)b_n} \leq (1+\delta)t^{-1}. \quad (3.4)$$

In a similar way for I_2 we have

$$0 \leq I_2 \leq (1+\delta)B([nt])([n\varepsilon]+1)^{-1} \sum_{k=[n\varepsilon]+1}^{[nt]} a_{n-k}.$$

Since $\sum_{k=[n\varepsilon]+1}^{[nt]} a_{n-k} = \sum_{k=0}^{n-[n\varepsilon]-1} a_k - \sum_{k=0}^{n-[nt]-1} a_k = o(L(n))$ ($n \rightarrow \infty$) we obtain

$$\lim_{n \rightarrow \infty} I_2/L(n)b_n = 0 \quad (3.5)$$

Now combine (3.2), (3.3) and (3.4). Let $\delta \downarrow 0$, $\varepsilon \downarrow 0$, $t \uparrow 1$ to obtain (ii).

Part 2: $0 < \alpha \leq 1$. First note that (i) is equivalent to

$$a_n \sim \alpha n^{\alpha-1} L(n) \quad (n \rightarrow \infty) \quad (3.6)$$

As in part 1 divide c_n into the three parts I_1 , I_2 and I_3 . As in part 1 we obtain

$$0 \leq \limsup_{n \rightarrow \infty} \frac{I_1}{na_n b_n} \leq \frac{(1-\varepsilon)^{\alpha-1} \varepsilon^{\beta+1}}{\beta+1}. \quad (3.7)$$

Also (3.3) remains valid, and we have

$$(1-\delta)t^{\beta+1} \frac{(1-t)^\alpha}{\alpha} \leq \lim_{n \rightarrow \infty} \left[\sup \right] \frac{I_3}{na_n b_n} \leq (1+\delta) \frac{(1-t)^\alpha}{\alpha t}. \quad (3.8)$$

As for I_2 we have ($[x]$ denotes the integral part of x)

$$I_2 = \int_{n-[nt]}^{n-[n\varepsilon]-1} a_{[s]} b_{n-[s]} ds = \int_{(n-[nt])/n}^{(n-[n\varepsilon]-1)/n} na_{[ns]} b_{n-[ns]} ds.$$

Using the uniform convergence properties of regularly varying sequences and functions (see e.g. [2], [13]) we obtain

$$\lim_{n \rightarrow \infty} \frac{I_2}{na_n b_n} = \int_{1-t}^{1-\varepsilon} s^{\alpha-1} (1-s)^\beta ds. \quad (3.9)$$

Now combine (3.7), (3.8) and (3.9). Let $\delta \downarrow 0$, $\varepsilon \downarrow 0$, $t \uparrow 1$ to obtain

$$\lim_{n \rightarrow \infty} \frac{c_n}{na_n b_n} = \int_0^1 (1-s)^\beta s^{\alpha-1} ds.$$

Using (3.6), the result follows. \square

3.2. Proof of Lemma 2.3. We only have to prove the result in case $\alpha \leq 0$ since the other case follows from Lemma 2.1. Let us write

$$c_n = \sum_{k=0}^{[n/2]} a_{n-k} b_k + \sum_{k=[n/2]+1}^n a_{n-k} b_k =: I_1 + I_2.$$

Since $\{a_n\}_{\mathbf{N}} \in RV_\alpha$ we have

$$\sup_{0 \leq k \leq [n/2]} a_{n-k}/a_n \leq K < \infty.$$

Hence Lebesgue's theorem yields $\lim_{n \rightarrow \infty} I_1/a_n = \sum_{k=0}^{\infty} b_k$ since for fixed k , $a_{n-k} \sim a_n$ ($n \rightarrow \infty$).

Next consider I_2 ; since $\{a_n\}_{\mathbf{N}} \in RV_\alpha$ we have [13, p. 20] for any $\varepsilon > 0$,

$$\sup_{k \leq n} (k^{-\alpha+\varepsilon} a_k) \sim n^{-\alpha+\varepsilon} a_n \quad (n \rightarrow \infty).$$

Hence for k such that $[n/2] + 1 \leq k \leq n - N$ and n, N large, we have

$$a_{n-k} \leq 2N^{\alpha-\varepsilon} n^{-\alpha+\varepsilon} a_n.$$

Hence for all k such that $[n/2] + 1 \leq k \leq n$ we have

$$a_{n-k} \leq M n^{-\alpha+\varepsilon} a_n \quad (3.10)$$

for some M and n sufficiently large. But then

$$0 \leq \frac{|I_2|}{a_n} \leq M' \sum_{k=[n/2]+1}^n k^{-\alpha+\varepsilon} |b_k| \rightarrow 0 \quad (n \rightarrow \infty).$$

This proves the lemma. \square

3.3. Proof of Theorem 2.4. and Corollary 2.5.

3.3.1. PROOF OF THEOREM 2.4. Let us write

$$\frac{c_n}{a_n} - \sum_{k=0}^{\infty} b_k = \sum_{k=1}^{\lfloor n/2 \rfloor} b_k \left(\frac{a_{n-k}}{a_n} - 1 \right) + \sum_{k=\lfloor n/2 \rfloor+1}^n b_k \frac{a_{n-k}}{a_n} - \sum_{k=\lfloor n/2 \rfloor+1}^{\infty} b_k =: I_1 + I_2 + I_3.$$

Since $n|I_3| \leq n \sum_{k=\lfloor n/2 \rfloor}^{\infty} |b_k| \leq 2 \sum_{k=\lfloor n/2 \rfloor}^{\infty} k|b_k|$ we have

$$\lim_{n \rightarrow \infty} nI_3 = 0. \quad (3.11)$$

Next consider I_2 . In case $\beta < 0$, from (2.1) we have $a_{n-1} \leq a_n$ for $n \geq N$. Hence for $n - k \geq N$ we have $a_N \leq a_{n-k} \leq a_n$. It follows that

$$0 \leq n|I_2| \leq \left(\left(\max_{0 \leq k \leq N} a_k \right) / a_N + 1 \right) n \sum_{k=\lfloor n/2 \rfloor}^n |b_k| \leq M \sum_{k=\lfloor n/2 \rfloor}^{\infty} k|b_k|$$

so that

$$\lim_{n \rightarrow \infty} nI_2 = 0. \quad (3.12)$$

In case $\beta \geq 0$, from (2.1) it follows that $\{a_n\}_{\mathbf{N}} \in RV_{-\beta}$. Using (3.10) we obtain

$$0 \leq n|I_2| \leq M n^{\beta+\varepsilon+1} \sum_{k=\lfloor n/2 \rfloor}^{\infty} |b_k| \leq M' \sum_{k=\lfloor n/2 \rfloor}^{\infty} k^{\beta+\varepsilon+1} |b_k|$$

so that (3.12) is also valid here.

Finally for I_1 note that (2.1) implies that for some M and all $n \geq 1$,

$$|a_{n-1}/a_n - 1| \leq M/n.$$

From [1, lemma 2] it follows that for $1 \leq k \leq \lfloor n/2 \rfloor$ and n large enough,

$$n|a_{n-k}/a_n - 1| \leq nk/(n-k)M' \leq kM''.$$

Hence in I_1 we can apply Lebesgue's theorem giving

$$\lim_{n \rightarrow \infty} nI_1 = \sum_{k=1}^{\infty} \lim_{n \rightarrow \infty} n \frac{a_{n-k} - a_n}{a_n} b_k = \beta \sum_{k=1}^{\infty} k b_k. \quad (3.13)$$

Combining (3.11) – (3.13) gives the desired result. \square

3.3.2. PROOF OF COROLLARY 2.5. For a sequence $\{d_n\}_{\mathbf{N}}$ denote by $D(z) = \sum_{n=0}^{\infty} d_n z^n$ its generating function. Then we have for $k = 0, 1, \dots$

$$R_{k+1}(z) = (R_k(1) - R_k(z))/(1-z), \quad P_{k+1}(z) = (z-1)P_k(z) + p_{k,k}z^k.$$

By induction it follows that for $k \geq 1$,

$$C(z) - \sum_{m=0}^{k-1} P_m(z)R_m(1) + \sum_{m=0}^{k-1} p_{m,m}z^m R_{m+1}(z) = P_k(z)R_k(z). \quad (3.14)$$

Now let us consider $\sum_{n=1}^{\infty} n^{\delta+1}|r_{m,n}|$ for $m = 0, 1, \dots, k$ and some $\delta \geq 0$. From the definition of $\{r_{m,n}\}_{\mathbf{N}}$ it follows that

$$\sum_{n=1}^{\infty} n^{\delta+1}|r_{m,n}| \leq \sum_{n=1}^{\infty} n^{\delta+1} \sum_{q=n+1}^{\infty} |r_{m-1,q}| = \sum_{q=2}^{\infty} \sum_{n=1}^{q-1} n^{\delta+1}|r_{m-1,q}|$$

so that $\sum_{n=1}^{\infty} n^{\delta+1}|r_{m,n}| \leq \sum_{n=1}^{\infty} n^{\delta+2}|r_{m-1,n}|$.

By induction it follows that

$$\sum_{n=1}^{\infty} n^{\delta+1}|r_{m,n}| \leq \sum_{n=1}^{\infty} n^{\delta+m+1}|r_{0,n}| = \sum_{n=1}^{\infty} n^{\delta+m+1}|b_n|.$$

Now $\sum_{n=1}^{\infty} n^{k+1}|b_n| < \infty$ and in case $\beta \geq 0$, $\sum_{n=1}^{\infty} n^{k+1+\beta+\varepsilon}|b_n| < \infty$ for some $\varepsilon > 0$. Hence for $m = 0, 1, \dots, k$,

$$\sum_{n=1}^{\infty} n|r_{m,n}| < \infty \text{ and in case } \beta \geq 0, \sum_{n=1}^{\infty} n^{\beta+\varepsilon+1}|r_{m,n}| < \infty. \quad (3.15)$$

Hence (2.4) will follow from (2.3), (3.14) and Theorem 2.4 if we can prove that for $m = 0, 1, \dots, k$

$$\lim_{n \rightarrow \infty} nr_{m,n}/p_{k,n} = 0. \quad (3.16)$$

Since $\{p_{k,n}\}_{\mathbf{N}} \in RV_{-\beta}$, in case $\beta < 0$, (3.16) follows at once from (3.15) and $p_{k,n} \rightarrow \infty$ ($n \rightarrow \infty$).

In case $\beta \geq 0$ we write for $\varepsilon > 0$,

$$\frac{nr_{m,n}}{p_{k,n}} = \frac{n^{\beta+\varepsilon+1}r_{m,n}}{n^{\beta+\varepsilon}p_{k,n}}.$$

Since $n^{\beta+\varepsilon}p_{k,n} \rightarrow \infty$ ($n \rightarrow \infty$), again (3.16) follows from (3.15). This completes the proof of the corollary. \square

3.4. Proof of the Theorem 2.6 and Corollary 2.8.

3.4.1. PROOF OF THEOREM 2.6. (i). For $n \geq 2N$ we have

$$\begin{aligned} |c_n| &\leq \sum_{k=N}^{n-N-1} \left| \frac{b_{n-k}}{a_{n-k}} \right| |a_{n-k}| |a_k| + \sum_{k=n-N}^n |b_{n-k}| |a_k| \\ &\leq \sup_{k \geq N} \left| \frac{b_k}{a_k} \right| \sum_{k=0}^n |a_{n-k}| |a_k| + \sup_{n-N \leq k \leq n} |a_k| \cdot \sum_0^{\infty} |b_k|. \end{aligned}$$

For n and k such that $N \leq n-k$, using (P1), it follows that $|a_{n-k}/a_n| \leq m^k$. Using (P2) it follows that $\sup_{n \geq 2N} |c_n|/|a_n| < \infty$ and hence also that $\sup_{n \geq N} |c_n/a_n| < \infty$. \square

3.4.2. PROOF OF THEOREM 2.6 (ii). For some fixed R and $n \geq R \geq N$ we have

$$\begin{aligned} \left| \frac{c_n}{a_n} - \sum_{k=0}^{\infty} b_k \right| &\leq \sum_{k=0}^R |b_k| \left| \frac{a_{n-k}}{a_n} - 1 \right| + \sum_{k=R+1}^n \left| \frac{b_k}{a_k} \right| \frac{|a_{n-k}| |a_k|}{|a_n|} + \sum_{k=R+1}^{\infty} |b_k| \\ &\leq \sum_{k=0}^R |b_k| \left| \frac{a_{n-k}}{a_n} - 1 \right| + \sup_{k \geq R} \left| \frac{b_k}{a_k} \right| \cdot M + \sum_{k=R+1}^{\infty} |b_k|. \end{aligned}$$

For fixed R , using (P3) we have

$$0 \leq \limsup_{n \rightarrow \infty} \left| \frac{c_n}{a_n} - \sum_{k=0}^{\infty} b_k \right| \leq \sup_{k \geq R} \left| \frac{b_k}{a_k} \right| \cdot M + \sum_{k=R+1}^{\infty} |b_k|.$$

Now let $R \rightarrow \infty$ to obtain (ii). \square

3.4.3. PROOF OF THEOREM 2.6 (iii). Since $b = \alpha a + d$ for some sequence d such that $\lim_{n \rightarrow \infty} d_n/a_n = 0$ we have $c = a * b = \alpha a * a + a * d$. Using (P4) and § 3.4.2 we obtain

$$\lim_{n \rightarrow \infty} \frac{c_n}{a_n} = 2\alpha \sum_{k=0}^{\infty} a_k + \sum_{k=0}^{\infty} d_k = \sum_{k=0}^{\infty} b_k + \alpha \sum_{k=0}^{\infty} a_k. \quad \square$$

3.4.4. PROOF OF COROLLARY 2.8. As in the proof of Corollary 2.5 we have

$$C(z) - \sum_{m=0}^{k-1} P_m(z) R_m(1) + \sum_{m=0}^{k-1} p_{m,m} z^m R_{m+1}(z) = P_k(z) R_k(z). \quad (3.17)$$

Now, if $\sup_{n \geq N} |r_{k,n}/p_{k,n}| < \infty$, by induction for $m = 0, 1, 2, \dots, k-1$, using (P1), we have

$$\sup_{n \geq N} \left| \frac{r_{m,n}}{p_{k,n}} \right| \leq \sup_{n \geq N} \left| \frac{r_{m+1,n-1}}{p_{k,n}} \right| + \sup_{n \geq N} \left| \frac{r_{m+1,n}}{p_{k,n}} \right| < \infty.$$

which using Theorem 2.6 (i) proves (i).

If $\lim_{n \rightarrow \infty} r_{k,n}/p_{k,n} = \alpha$, by induction for $m = 0, 1, 2, \dots, k-1$ using (P3), we have

$$\lim_{n \rightarrow \infty} \frac{r_{m,n}}{p_{k,n}} = \lim_{n \rightarrow \infty} \frac{r_{m+1,n-1} - r_{m+1,n}}{p_{k,n}} = 0. \quad (3.18)$$

From (3.17), (3.18) and Theorem 2.6 it follows that

$$\lim_{n \rightarrow \infty} \left(c_n - \sum_{m=0}^{k-1} p_{m,n} R_m(1) \right) / p_{k,n} = -\alpha p_{k-1,k-1} + \alpha \sum_{m=0}^{\infty} p_{k,m} + R_k(1).$$

Since $\sum_{m=0}^{\infty} p_{k,m} = p_{k-1,k-1}$ the proof of (ii) is complete. \square

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