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ON EDGE-COLORABILITY OF PRODUCTS OF GRAPHS

Bojan Mohar

Abstract. Let $\chi'(G)$ denote the edge-chromatic number and $\Delta(G)$ the maximum vertex degree of a graph G. A graph G is said to be *of class* 1 if $\chi'(G) = \Delta(G)$ and *of class* 2 otherwise. Some sufficient conditions for various graph products (the Cartesian, lexicographic, tensor and strong product) to be of class 1 are given.

1. Introduction and definitions. This note extends results of Himmelwright and Wiliamson [3], Kotzig [4], Mahmoodian [5] and Mohar, Pisanski and Shawe-Taylor [6, 7] concerning the edge-colorability of various graph products. Up till now we considered only the products of regular graphs, being inspired by the work of Kotzig [4]. The paper [6] summarizes our work in the regular case. However, most of the results can be extended to the products of non-regular graphs with almost no extra effort. In some cases, though, the condition of regularity is essential.

We will leave the basic definitions of graph theory to a standard reference book [2], and will limit ourselves to defining only less known terms and those wish may cause confusion.

Let v and u be vertices of a graph G. We write $v \sim u$ to denote that v and u are adjacent. Let $\chi'(G)$ denote the edge-chromatic number and $\Delta(G)$ the maximum vertex degree of a graph G. By the well-known Theorem of Vizing [9] on edge-colorability of graphs we can classify graphs into two classes. A graphs G is said to be *of class* 1 if $\chi'(G) = \Delta(G)$, and *of class* 2 if $\chi'(G) = \Delta(G) + 1$.

The *Cartesian product* $G \times H$ of graphs G and H has vertex set $V(G) \times V(H)$ and edge set

$$E(G \times H) = \{(u,v)(u',v');\ \text{ either } (u = u' \text{ and } v \sim v') \text{ or } (u \sim u' \text{ and } v = v')\}.$$

The *lexicographic product* $G \circ H$ of graphs G and H has vertex set $V(G) \times V(H)$ and edges $E(G \circ H) = \{(u,v)(u',v');$ either $u \sim u'$ or $(u = u'$ and $v \sim v')\}$. Note that $G \circ H$ is often written as $G[H]$.

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The *tensor product* of graphs G and H is defined as the graph $G \otimes H$ with vertex set $V(G) \times V(H)$ and the edge set

$$E(G \times H) = \{(u,v)(u',v');\ u \sim u' \text{ and } v \sim v'\}.$$

If G and H have the same vertex set $V = V(G) = V(H)$, and disjoint edge sets, $E(G) \cap E(H) = \emptyset$, then their *sum* $G \oplus H$ is defined as the graph having the edge set $E(G \oplus H) = E(G) \cup E(H)$. We also say that G and H are *factors* of $G \oplus H$.

The *strong product* $G * H$ is defined as $G * H = (G \circ H) \oplus (G \times H)$.

All defined products are associative thus the products of more than two graphs can be defined without confusion.

2. Results. First of all we state a simple lemma, whose trivial proof we omit.

2.1. LEMMA. *Let F_1 and F_2 be two graphs which are of class 1 and let $F = F_1 \oplus F_2$. If*

$$\Delta(F) = \Delta(F_1) + \Delta(F_2) \tag{1}$$

then F is of class 1.

Now we present the generalization of Kotzig's result [4] for the cartesian product of graphs.

2.2. THEOREM. *Let H be the cartesian product of graphs $G_1, G_2, \dots, G_n$ and let one of the following two conditions be satisfied*

(i) *at least one of the graphs G_i is nontrivial and of class 1,*

(ii) *there exist at least two distinct indices i and j, such that G_i and G_j both contain a 1-factor.*

Then H is of class 1.

It is worth noting that the sufficient condition (i) of Theorem 2.2 was obtained before by several authors. It was first given as an exercise in Bondy and Murty's book [1, exercise 6.2.6], then obtained by Himmelwright and Williamson [3] for the case of regular graphs and letter extended by Mahmoodian [5] to the general case. At last this condition appeared in Kotzig's paper [4]. Condition (ii) can also be found there, though it is given only for the case of regular graphs. A proof similar to that in [4] can be applied to the nonregular case: Let $G_i = F_i \oplus H_i$ and $G_j = F_j \oplus H_j$, where F_i and F_j are 1-factors. Then $G_i \times G_j = (F_i \oplus H_i) \times (F_j \oplus H_j) = (F_i \times H_j) \oplus (H_i \times F_j)$. By (i), both graphs $F_i \times H_j$ and $H_i \times F_j$ are of class 1 as F_i and F_j are of class 1. It is easy to see that the factorization $(F_i \times H_j) \oplus (H_i \times F_j)$ satisfies condition (1) of Lemma 2.1. We conclude that $G_i \times G_j$ is of class 1. Now, part (i) of the theorem applies to the general case.

Let us now consider the tensor and strong products.

2.3 THEOREM. *Let K be the tensor and H the strong product of graphs $G_1, G_2, \dots, G_n$ and let at least one of the graphs G_i be nontrivial and of class 1. Then K and H are of class 1.*

We will omit the proof of Theorem 2.3, as it is the same as the one given in [7] for the regular case. Again, condition (1) of Lemma 2.1 is easily verified for all the factorizations used in the proof.

2.4. Theorem. *The lexicographic product* $K = G_1 \circ G_2 \ldots \circ G_n$ *is of class* 1 *if at least one of the following conditions is satisfied:*

(i) G_1 *is a nontrivial graph which is of class* 1*,*

(ii) *for some* $i (2 \leq i \leq n)$ *the graph* G_i *is nontrivial and of class* 1 *and for some* $j \geq i$ *the graph* G_j *is of even order or*

(iii) *there exists two distinct indices* i *and* j*, such that the graphs* G_i *and* G_j *both contain a* 1*-factor.*

Before proving the theorem we need a lemma:

2.5. Lemma. *Let* H *be a lexicographic product of graphs* $F_1, F_2, \ldots, F_n$ *and assume that for some* $i (1 \leq i \leq n)$ *the graph* F_i *has a* 1*-factor. Then* H *has a* 1*-factor.*

Proof. By the law of associativity of the lexicographic product

$$H = (F_1 \circ \ldots \circ F_{i-1}) \circ F_i \circ (F_{i+1} \circ \ldots \circ F_n)$$

If follows that it suffices to prove the lemma only for the case $n = 2$. But since

$$F_1 \circ F_2 = (F_1 \times F_2) \oplus (F_1 \oplus K_m), \quad m = |V(F_2)|$$

and $F_1 \times F_2$ contains a 1-factor if either F_1 or F_2 does, the lemma follows.

Proof of Theorem 2.4. From

$$K = G_1 \circ (G_2 \circ \ldots \circ G_n) = (G_1 \circ \ldots \circ G_{i-1}) \circ (G_i \circ (G_{i+1} \circ \ldots \circ G_n))$$

we can conclude that it suffices to give a proof of (i) and (ii) only for the case $n = 2$. Since

$$G_i \circ \ldots \circ G_j = G_i \circ (G_{i+1} \circ \ldots \circ G_j)$$

and since by Lemma 2.5 $G_{i+1} \circ \ldots G_j$ has a 1-factor if G_j has a 1-factor, the same is true for the case (iii).

Let $K = G_1 \circ G_2$. The lexicographic product can be factored in an obvious way,

$$K = G_1 \circ G_2 = (G_1 \times G_2) \oplus (G_1 \oplus K_m), \;\; m = |V(G_2)|. \tag{2}$$

By Theorem 2.2, the cartesian product $G_1 \times G_2$ is of class 1 if at least one of the conditions (i)–(iii) is satisfied. The maximum degrees of graphs in the factorization (2) are:

$$\begin{aligned}
\Delta(G_1 \circ G_2) &= m\Delta(G_1) + \Delta(G_2) \\
\Delta(G_1 \times G_2) &= \Delta(G_1) + \Delta(G_2) \\
\Delta(G_1 \oplus K_m) &= \Delta(G_1) \cdot \Delta(K_m) = \Delta(G_1).
\end{aligned}$$

Therefore Lemma 2.1 can be applied and we have to prove only that the graph $G_1 \oplus K_m$ is of class 1. But this is obvious: in case (i) the graph G_i is of class 1, whereas in cases (ii) and (iii) m is even and thus K_m is of class 1 (existence of a 1-factor implies that G_2 has an even number of vertices). By Theorem 2.3 the tensor product $G_1 \oplus K_m$ is of class 1. This completes the proof.

The requirement in Theorem 2.4 (ii) that the graph $G_i \circ G_{i+1} \circ \ldots \circ G_n$ has an even number of vertices is essential. For example the graphs $C_3 \circ (K_1 \cup K_2)$ and $C_{2n+1} \circ P_{2k+1}$ $(n \leq k)$ are of class 2, though $K_1 \cup K_2$ and P_{2k+1} are of class 1. In fact there is a general construction of such graphs: Let G be any regular graph of odd order. Then for all $k's$ large enough, $G \circ P_{2k+1}$ is of class 2. To see this one observes that the given examples have only vertices of degree d and $d-1$. If the graph has an odd number of vertices, and, moreover, if the number of vertices of degree $d-1$ is less than d one can verify that such a graph must be of class 2.

3. Concluding remarks. It is interesting that Theorem 2.2 can be put in a more general setting. In particular, the results for the cartesian product can be seen as a special case of a similar result for graph bundles, obtained by Pisanski, Shawe-Taylor and Vrabec [8].

It is worth noting that the sufficient conditions of Theorems 2.2, 2.3 and 2.4 are not necessary, as it is shown in [4, 7] in the case of regular graphs.

Finally we mention some open problems concerning the classification of various graph products.

PROBLEM 1. *Let G and H both have 1-factors. Does it follow that $G \oplus H$ is of class 1?*

The answer to problem 1 is not known even for regular graphs.

PROBLEM 2. *Let H be a nontrivial graph of class 1 and of odd order and let G be of even order. Is the lexicographic product $G \circ H$ of class 1?*

PROBLEM 3. *Are the conditions* (i) *or* (ii) *of Theorem 2.2 sufficient for some more general graph products to be of class 1?*

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E. Kardelj University
Department of Mathematics
Jadranska 19, P.O.B. 64
61111 Ljubljana, Yugoslavia

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