

A HEREDITARY PROPERTY OF HM-SPACES

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Abstract. We have shown that a class of HM-spaces is invariant under projective topology, in particular, arbitrary product, subspace and separated quotient. We have shown also that $(E, \sigma(E, E'))$ is an HM-space for every locally convex space (E, t) (see [2, Theorem 513]).

The nonstandard theory of topological vector spaces and in particular the construction of the nonstandard hull of an arbitrary topological vector space has been studied by Henson and Moore in [2] and [3]. We recount the principal ideas and definitions.

Let (E, t) be a locally convex space and let ${}^*\mathcal{M}$ be a nonstandard extension of a superstructure \mathcal{M} which contains (E, t) . An element $p \in {}^*E$ is called t -finite if for every t -neighborhood U of zero there exists an integer n such that $x \in n{}^*U$ and the set of t -finite elements of *E is denoted by $\text{fin}_t({}^*E)$. The monad of O is defined by $\mu_t(O) = \bigcap_{U \in \mathcal{U}} {}^*U$, where \mathcal{U} is a base of balanced, convex neighborhood of zero.

Both $\text{fin}_t({}^*E)$ and $\mu_t(O)$ are vector spaces over the same field as E . We denote the quotient vector space $\text{fin}_t({}^*E)/\mathcal{M}_t(O)$ by \hat{E} , the canonical quotient mapping of $\text{fin}_t({}^*E)$ onto \hat{E} by π and the quotient topology defined on \hat{E} by \hat{t} . The nonstandard hull of (E, t) with respect to ${}^*\mathcal{M}$ is the separated quotient space (\hat{E}, \hat{t}) . Clearly, the map taking x to $\pi({}^*x)$ is a topological vector space isomorphism of (E, t) into (\hat{E}, \hat{t}) . An element $p \in {}^*E$ is called t -pre-near-standard if for each t -neighborhood U there exists $y \in E$ such that $x \in {}^*y + {}^*U$ and the set of t -pre-near-standard elements of *E is denoted by $\text{pns}_t({}^*E)$.

The nonstandard hull (\hat{E}, \hat{t}) of a locally convex space (E, t) always contains $\pi(\text{pns}_t({}^*E)) =$ the completion of (E, t) and the nonstandard hulls determined by different nonstandard extensions ${}^*\mathcal{M}$ need not even be isomorphic to each other. When every nonstandard hull (\hat{E}, \hat{t}) is equal to $\pi(\text{pns}_t({}^*E))$, we say that the nonstandard hulls of (E, t) are invariant, i.e. we say that a locally convex space (E, t) is an HM-space (see [4]).

Throughout this paper (E, t) will denote a separated locally convex space over K , where K denotes the real or complex numbers.

Henson and Moore have shown the following [3]:

THEOREM 1 (Henson and Moore). *The following conditions on a locally convex space (E, t) are equivalent:*

- (a) *The nonstandard hulls of (E, t) are invariant;*
- (b) *The nonstandard hull of (E, t) is isomorphic to the completion of (E, t) for every choice of the enlargement ${}^*\mathcal{M}$;*
- (c) *$\text{fin}_t({}^*E) = \text{pns}_t({}^*E)$ for every choice of ${}^*\mathcal{M}$;*
- (d) *The nonstandard hull of (E, t) is isomorphic to the completion of (E, t) for some choice of the enlargement ${}^*\mathcal{M}$;*
- (e) *$\text{fin}_t({}^*E) = \text{pns}_t({}^*E)$ for some choice of ${}^*\mathcal{M}$;*
- (f) *If \mathcal{F} is an ultrafilter on E and for each t -neighborhood U of zero there is an integer n such that $nU \in \mathcal{F}$, then \mathcal{F} is a Cauchy filter.*

For proofs of our theorems ([2], [3], [4] and [5]) we use a conditions (e) or (f) of preceding theorem.

THEOREM 2. *Every separated quotient $(E/H, t)$ of an HM-space (E, t) is an HM-space.*

Proof. Let ${}^*\mathcal{M}$ be a nonstandard extension of a superstructure \mathcal{M} which contains a space (E, t) . According to the condition (e) of the preceding theorem and [2, Theorem 1.2] it is sufficient to prove that $\text{fin}_t({}^*(E/H)) \subset \text{pns}_t({}^*(E/H))$. Let $x \in \text{fin}_t({}^*(E/H))$. Then, by [1, Definition 1.1] for each t -neighborhood U of zero there exists an integer n such that $x \in n^*(U + {}^*H) = n^*U + {}^*H$, i.e. $x \in \text{fin}_t({}^*E) + {}^*H$. By assumption (E, t) is an HM-space and then $x \in \text{pns}_t({}^*E) + {}^*H$. From this and [2, Theorem 1.2. iv] it follows that for every t -neighborhood U of zero there is $y \in E$ such that $x \in {}^*y + {}^*U + {}^*H = {}^*(y + H) + {}^*(U + H)$, i.e. $x \in \text{pns}_t({}^*(E/H))$. Hence, the separated quotient space $(E/H, t)$ of a HM-space (E, t) is an HM-space.

THEOREM 3. *A locally convex space (E, t) is an HM-space, if and only if all its subspaces are HM-spaces.*

Proof. The sufficiency follows immediately from [1, Theorem 4.6]. To prove the necessity we utilize condition (f) of Theorem 1. Let \mathcal{F} be an ultrafilter on its subspace (H, t_H) with the property that for each t_H -neighborhood U of zero, there exists an integer n such that $nU \in \mathcal{F}$ (t_H is the relative topology in H). By [1, chapter I (6), proposition 10] \mathcal{F} is the base of an ultrafilter on the space (E, t) such that for every t -neighborhood V of zero, there exists an integer n such that $nV \in \mathcal{F}$ ($V \supset V \cap H$ and for some: $n(V \cap H) \in \mathcal{F}$ i.e. $nv \in \mathcal{F}$). By assumption (E, t) is an HM-space and then \mathcal{F} is a base of Cauchy filter according to Theorem 1 (f). We want to show that \mathcal{F} is a Cauchy filter on the space (H, t_H) . If V

is a t_H -neighborhood of zero, there exists a t -neighborhood U of zero such that $V \supset U \cap H$. For t -neighborhood U of zero there exists $A \in \mathcal{F}$ such that $A - A \subset U$, i.e. $A \cap H - A \cap H \subset A - A \subset U$ and then $A \cap H - A \cap H \subset U \cap H \subset V$. Hence, for each t_H -neighborhood V of zero there exists $B \in \mathcal{F}$ ($B = A \cap H$) such that $B - B \subset V$, i.e. \mathcal{F} is a Cauchy filter on the space (H, t_H) , so the condition is necessary.

THEOREM 4. *The topological product of a family (E_i, t_i) $i \in I$ of HM-spaces is an HM-space if and only if every space (E_i, t_i) is an HM-space.*

Proof. The necessity follows from the preceding theorem. To prove the sufficiency we use condition (e) of Theorem 1. Clearly, it is sufficient to prove that $\text{fin}_t^*(\prod_{i \in I} E_i) \subset \text{pns}_t^*(\prod_{i \in I} E_i)$, where t is a product topology. Let $x \in \text{fin}_t^*(\prod_{i \in I} E_i)$ and let $U = \prod_{i=1}^n V_1 \times \prod_{i \neq 1, 2, \dots, n} E_i$ be a neighborhood of zero of the space $(\prod_{i \in I} E_i, t_i)$. By [2, Definition 1.1] there exists an integer n such that $x \in n^*U = n^*\left(\prod_{i=1}^n V_1 \times \prod_{i \neq 1, 2, \dots, n} E_i\right)$. According to transfer principle it follows that $x(i) \in n^*V_i$ for $i \in \{1, 2, \dots, n\}$ and $x(i) \in {}^*E_i$ for $i \neq 1, 2, \dots, n$. Therefore, $x(i) \in \text{fin}_{t_i}({}^*E_i) = \text{pns}_{t_i}({}^*E_i)$ for $i \in \{1, 2, \dots, n\}$, i.e. for neighborhoods V_i there exists $y(i) \in E_i$ such that $x(i) \in {}^*y(i) + {}^*V_i$ and $y(i) = O \in E_i$ for $i \neq 1, 2, \dots, n$. Hence, for each t -neighborhood U of zero of the space $(\prod_{i \in I} E_i, t)$ there exists $y \in \prod_{i \in I} E_i$ such that $x \in {}^*y + {}^*U$, i.e. $x \in \text{pns}_t^*\left(\prod_{i \in I} E_i\right)$.

THEOREM 5. *Let (E_i, t_i) , $i \in I$, be a family of HM-spaces. Then the linear space E equipped with a locally convex separated topology t is an HM-space, if t is the projective topology for t_i , $i \in I$.*

Proof. Let \mathcal{F} be an ultrafilter on E with the property that for each t -neighborhood V of zero, there exists an integer n such that $nV \in \mathcal{F}$. By [1, Chapter I(6), Proposition 10] every $f_i(\mathcal{F})$ is a base of ultrafilter for which the condition (f) of Theorem 1 holds. But, locally convex spaces (E_i, t_i) are HM-spaces and therefore for every t_i -neighborhood V_i of zero there exists $A \in \mathcal{F}$ such that $A - A \subset f_i^{-1}(f_i(A - A)) \subset f_i^{-1}(V_i)$, so $A - A \subset \bigcap_{i=1}^n f_i^{-1}(V_i)$. Hence, \mathcal{F} is a Cauchy filter on the space (E, t) and according to Theorem 1 (f), the theorem is proved.

COROLLARY 1. *The projective limit of any family of HM-spaces is an HM-space.*

Proof. The proof follows Theorem 5 and [5, chapter II.5].

COROLLARY 2. *If (E, t_i) , $i \in I$ is any family of a HM-spaces, then $(E, \sup t_i)_{i \in I}$ is an HM-space.*

Proof. A topology $\sup t_i$, $i \in I$ is projective for the systems (E, t_i) and then the proof follows by the preceding theorem.

COROLLARY 3. *For every locally convex space (E, t) , the associated space $(E, \sigma(E, E'))$ is an HM-space.*

Proof. The weak topology is projective for the systems $(K_i, t_i)_{i \in E'}$, where K is the field of the real or complex numbers and t_i is the usual euclidean topology for every $i \in E'$ (E' is the vector space of continuous linear functionals on a locally convex space (E, t)). Hence, by the preceding theorem the space $(E, \sigma(E, E'))$ is an HM-space.

COROLLARY 4 [5, Chapter IV, (3.3)]. *The space (E, τ_m) is an HM-space, if τ_m , is the minimal locally convex topology on E .*

Proof. According to [5] $\tau_m = \sigma((E')^\#, E')$ and by Corollary 3 (E, τ_m) is an HM-space.

Remark 1. By [3, Theorem 2] and [5, Chapter II. 6, Example 1] it follows that the class of HM-spaces is not invariant under direct sums and inductive limit.

Remark 2. According to Corollary 3, it is easy to see that the barrelled (quasi-barrelled, bornological, ultra-bornological) space associated to an HM-space is not an HM-space, in general. (About associated spaces see [6]).

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