

**SOME SPECIAL PRODUCT SEMISYMMETRIC
AND SOME SPECIAL
HOLOMORPHICALLY SEMISYMMETRIC F-CONNECTIONS**

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Abstract. In the present paper we investigate two special product semisymmetric F-connections: PS-concircular and PS-coharmonic ones and we find the conditions for product semisymmetric connection to be PS-concircular or PS-coharmonic. In the same manner we investigate two special holomorphically semisymmetric F-connections.

Introduction. The connection of an n -dimensional differentiable manifold is termed semisymmetric if its torsion tensor S satisfies

$$(0.1) \quad S_{jk}^i = \delta_j^i S_k - \delta_k^j S_j.$$

A semisymmetric connection is generalized on a locally decomposable Riemannian space in [4] and [5]. In [4] the product semisymmetric metric F-connection is defined and studied. In [5] something similar is done for the holomorphically semisymmetric F-connection.

In the present paper we investigate two special holomorphically semisymmetric and two special product semisymmetric F-connections. In **1** we recall what is a product semisymmetric F-connection. In **2** we define PS-concircular connections and prove that each of the relations (2.7), (2.9) and (2.17) is a necessary and sufficient condition for a product semisymmetric connection to be PS-concircular. In **3** we define PS-coharmonic connections and prove that a product semisymmetric connection is PS-coharmonic iff (3.3) holds. In **4** we recall what is a holomorphically semisymmetric connection. In **5** we define HS-concircular connection and prove that each of the relations (5.6) and (5.8) is a necessary and sufficient condition for a holomorphically semisymmetric connection to be HS-concircular. In **6** we investigate another HS-connection.

These results generalize, for the locally decomposable Riemannian space, the results obtained by P. Strave in [6].

1. Product semisymmetric metric F-connection. An n -dimensional differentiable manifold M_n of class C^∞ is called a locally decomposable Riemannian space [8] if in M^n a tensor field $F_j^i \neq \delta_j^i$ and a positive definite Riemannian metric $ds^2 = g_{ij}(x^k)dx^i dx^j$ are given, satisfying the conditions

$$(1.1) \quad F_k^i F_j^k = \delta_j^i, \quad g_{ab} F_i^a F_j^a = g_{ij}, \quad \nabla_k F_j^i = 0,$$

where ∇_k is the operator of the covariant derivative with respect to the Riemannian metric. If we put $F_i^a g_{aj} = F_{ij}$, then $F_{ij} = F_{ji}$ and the condition $\nabla_k F_j^i = 0$ is equivalent to the condition $\nabla_k F_{ij} = 0$.

Locally decomposable space can be covered by a separating coordinate system, that is, by such a system of coordinate neighborhoods (x^i) that in any intersection of two coordinate neighborhoods (x^i) and $(x^{i'})$ we get

$$x^{a'} = x^{a'}(x^a), \quad x^{y'} = x^{y'}(x^y),$$

where the indices a, b, c run over the range $1, 2, \dots, p$ and the indices x, y, z run over the range $p+1, \dots, p+q = n$.

With respect to a separating coordinate system, the metric of the space has the form $ds^2 = g_{ab}(x^c)dx^a dx^b + g_{xy}(x^z)dx^x dx^y$, while the tensors F_{ij} and F_j^i have the forms:

$$(F_{ij}) = \begin{pmatrix} g_{ab} & 0 \\ 0 & -g_{xy} \end{pmatrix}, \quad \begin{pmatrix} \delta_a^b & 0 \\ 0 & -\delta_y^x \end{pmatrix}.$$

Therefore

$$(1.2) \quad \varphi \equiv p - q.$$

In the following we suppose $p > 2, q > 2$.

The product semisymmetric metric F-connection (PS-connection) has the form [4]:

$$(1.3) \quad \Gamma_{jk}^i = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} + \delta_j^i S_k - g_{jk} S^i + F_j^i S_a F_k^a - S^a F_a^i F_{jk},$$

where S_i is a decomposable vector field in M_n , i.e. the field satisfying the condition

$$(1.4) \quad F_j^a \nabla_a S + i = F_i^a \nabla_j S_a,$$

and $S^i = g^{ia} S_a$. This condition can be expressed in the form

$$(1.5) \quad F_i^a F_j^b \nabla_b S_a = \nabla_j S_i.$$

Also, we suppose in the following that S_i is locally a gradient vector field.

With respect to a separating coordinate system, all the Γ_{jk}^i are zero except

$$\Gamma_{bc}^a = \left\{ \begin{matrix} a \\ bc \end{matrix} \right\} + 2S_c \delta_b^a - 2g_{bc} S^a, \quad \Gamma_{yz}^x = \left\{ \begin{matrix} x \\ yz \end{matrix} \right\} + 2S_z \delta_y^x - 2g_{zy} S^x,$$

that is, both connections Γ_{bc}^a and Γ_{yz}^x are semisymmetric metric connections: the first with respect to the metric $g_{ab}(x^c)dx^a dx^b$, the second with respect to the metric $g_{xy}(x^z)dx^x dx^y$.

The curvature tensor R_{rkj}^i of the connection (1.3) can be expressed in the form [4]

$$(1.6) \quad R_{rkj}^i = K_{rkj}^i + \delta_j^i \rho_{rk} - g_{jr} \rho_k^i + F_j^i F_r^a \rho_{ak} - F_{jr} g^{ib} F_b^a \rho_{ak} - \\ - \delta_k^i \rho_{rj} + g_{kr} \rho_j^i - F_k^i F_r^a \rho_{aj} + F_{kr} g^{ib} F_b^a \rho_{aj},$$

where

$$(1.7) \quad \rho_{rk} = \nabla_r S_k - S_r S_k + S^a S_a g_{rk} / 2 - S_a S_b F_r^a F_k^b + S^a S_b F_a^b F_{rk} / 2,$$

and K_{rkj}^i is the curvature tensor of the Riemannian space having g_{ij} as a metric tensor. Since S_i is a gradient, tensor ρ_{rk} is symmetric. Also, it satisfies the condition

$$(1.8) \quad \rho_{ab} F_i^a F_j^b = \rho_{ij}.$$

because of (1.5). If we put

$$R_{rk} = R_{rka}^a, \quad R_{rk}^* = R_{ra} F_k^a, \quad K_{rk} = K_{rka}^a, \quad K_{rk}^* = K_{ra} F_k^a,$$

we obtain from (1.6), using (1.8)

$$(1.9) \quad R_{rk} = K_{rk} + (n-4)\rho_{rk} + \varphi F_r^a \rho_{ak} + g_{rk} \rho_a^a + F_{rk} F_b^a \rho_a^b,$$

$$(1.10) \quad R_{rk}^* = K_{rk}^* + (n-4)\rho_{ra} F_k^a + \varphi \rho_{rk} + F_{rk} \rho_a^a + g_{rk} F_b^a \rho_a^b.$$

The Ricci tensor K_{ij} of the locally decomposable Riemannian space is a pure tensor, i.e. $K_{ab} F_i^a F_j^b = K_{ij}$, from which $K_{aj} F_i^a = K_{ia} F_j^a$, i.e. K_{ij}^* is a symmetric tensor. The tensor $\rho_{ak} F_r^a$ is a symmetric tensor too, because of (1.8). Therefore, both R_{rk} and R_{rk}^* are symmetric tensors.

We obtain from (1.6)

$$R_{irkj} = K_{irkj} + g_{ij} \rho_{rk} - g_{jr} \rho_{ik} + F_{ij} F_r^a \rho_{ak} - F_{jr} F_i^a \rho_{ak} - \\ + g_{ik} \rho_{rj} - F g_{kr} \rho_j^i + F_{ik} \rho_{aj} + F_{kr} F_i^a \rho_{aj},$$

and see that

$$(1.11) \quad R_{rikj} = -R_{irkj}.$$

Eliminating ρ_{ij} from (1.6), we obtain [4]

$$R_{rkj}^i + b_1 [s_{rkj}^{-i} - 2(bR - aR^*) r_{rkj}^i + 2(aR - bR^*) r_{akj}^i F_r^a] - \\ - a_1 [s_{akj}^{-i} F_r^a - 2(bR - aR^*) r_{akj}^i F_r^a + 2(aR - bR^*) r_{rkj}^i] = \\ = K_{rkj}^i + b_1 [s_{rkj}^i - 2(bK - aK^*) r_{rkj}^i + 2(aK - bK^*) r_{akj}^i F_r^a] - \\ - a_1 [s_{akj}^i F_r^a - 2(bK - aK^*) r_{akj}^i F_r^a + 2(aK - bK^*) r_{rkj}^i],$$

where we have put

$$\begin{aligned} R &= R_a^a, \quad K = K_a^a, \quad R^* = R_a^{*a}, \quad K^* = L_a^{*a}, \\ s_{rkj}^{-i} &= \delta_j^i R_{rk} - g_{rj} R_k^i + F_j^i R_{rk}^* - F_{rj} R_k^{*i} - \\ &\quad - \delta_k^i R_{rj} + g_{rk} R_j^i - F_k^i R_{rj}^* + F_{rk} R_j^{*i}, \end{aligned} \quad (1.13)$$

$$\begin{aligned} s_{rkj}^i &= \delta_j^i K_{rk} - g_{rj} K_k^i + F_j^i K_{rk}^* - F_{rj} K_k^{*i} - \\ &\quad - \delta_k^i K_{rj} + g_{rk} K_j^i - F_k^i K_{rj}^* + F_{rk} K_j^{*i} \\ r_{rk}^i &= \delta_k^i g_{rj} - \delta_j^i g_{rk} + F_k^i F_{rj} - F_j^i F_{rk} \end{aligned} \quad (1.14)$$

$$\begin{aligned} a &= \frac{\varphi}{2[\varphi^2 - (n-2)^2]}, \quad b = \frac{n-2}{2[\varphi - (n-2)^2]}, \\ a_1 &= \frac{\varphi}{\varphi^2 - (n-4)^2}, \quad b_1 = \frac{n-2}{\varphi^2 - (n-4)^2}. \end{aligned} \quad (1.15)$$

2. Product semisymmetric concircular connection. In this section we consider a special product semisymmetric metric F-connection, namely the connection (1.3) satisfying the condition

$$(2.1) \quad \rho_{ik} = f g_{ik} + h F_{ik},$$

where f and h are some scalar functions. From (1.6) and (2.1) it results that

$$\begin{aligned} R_{rkj}^i &= K_{rkj}^i + 2f(\delta_j^i g_{kr} - \delta_k^i g_{jr} + F_j^i F_{kr} - F_k^i F_{jr}) + \\ &\quad + 2h(\delta_j^i F_{kr} - \delta_k^i F_{jr} + F_j^i g_{kr} - F_k^i g_{jr}). \end{aligned} \quad (2.2)$$

Contracting (2.2) with respect to i and j we obtain

$$(2.3) \quad R_{rk} = K_{rk} + 2f[(n-2)g_{kr} + \varphi g_{kr}] + 2h((n-2)F_{kr} + \varphi g_{kr}).$$

It follows from (2.3) that

$$(2.4) \quad R_{rk}^* = K_{rk}^* + 2f[(n-2)F_{kr} + \varphi g_{kr}] + 2h((n-2)g_{kr} + \varphi F_{kr}).$$

Transvecting (2.3) and (2.4) with g^{rk} , we find that

$$\begin{aligned} R - K &= 2f[n(n-2) + \varphi^2] + 4h\varphi(n-1), \\ R^* - K^* &= 4f\varphi(n-1) + 2h[n(n-2) + \varphi^2]. \end{aligned}$$

Since $p > 2$ and $q > 2$, $[n(n-2) + \varphi^2]^2 - 4\varphi^2(n-1)^2 \neq 0$, and the above relations give

$$(2.5) \quad 2f = -\alpha(R - K) - \beta(R^* - K^*), \quad 2h = -\beta(R - K) - \alpha(R^* - K^*),$$

where

$$\alpha = -\frac{n(n-2) + \varphi^2}{[n(n-2) + \varphi^2]^2 - 4\varphi^2(n-1)^2}, \quad \beta = \frac{2\varphi(n-1)}{[n(n-2) + \varphi^2]^2 - 4\varphi^2(n-1)^2}.$$

Substituting (2.5) into (2.2) and taking into account the notation (1.14), we get

$$(2.7) \quad \begin{aligned} R_{rkj}^i - (\alpha R + \beta R^*)r_{rkj}^i - (\beta R + \alpha R^*)R_{akj}^i F_r^a = \\ = K_{rkj}^i - (\alpha K + \beta K^*)r_{rkj}^i - (\beta K + K^*)r_{akj}^i F_r^o. \end{aligned}$$

Conversely, we suppose that for (1.3) we have (2.7). Then, substituting $R_{rkj}^i - K_{rkj}^i$ from (2.7) into (1.6), we obtain

$$\begin{aligned} [\alpha(R - K) + \beta(R^* - K^*)]r_{rkj}^i + \beta[(R - K) + \alpha(R^* - K^*)]r_{akj}^i F_r^a = \\ = \delta_j^i \rho_{rk} - \delta_k^i \rho_{rj} - g_{jr} \rho_k^i + g_{kr} \rho_j^i + F_j^i F_r^a \rho_{ak} - F_k^i F_r^a - \\ F_{jr} g^{ib} F_b^a \rho_{ak} + F_{kr} g^{ib} F_b^a \rho_{aj}. \end{aligned}$$

Contracting this with respect to i and k and taking into account (1.8), we get

$$\begin{aligned} (4 - n)\rho_{rj} - \varphi F_r^a \rho_{aj} = \\ = \{(n - 2)[\alpha(R - K) + \beta(R^* - K^*)] + \varphi[\beta(R - K) + \alpha(R^* - K^*)] + \rho_a^a\}g_{jr} + \\ + \{\varphi[\alpha(R - K) + \beta(R^* - K^*)] + (n - 2)[\beta(R - K) + \alpha(R^* - K^*)] + F_b^a \rho_a^b\}F_{ij} \end{aligned}$$

or

$$(2.8) \quad (4 - n)\rho_{rj} - \varphi F_r^a \rho_{aj} = f_1 d_{jr} + h_1 F_{jr},$$

where we have put

$$\begin{aligned} f_1 &= (n - 2)[\alpha(R - K) + \beta(R^* - K^*)] + \varphi[\beta(R - K) + \alpha(R^* - K^*)] + \rho_a^a, \\ h_1 &= \varphi[\alpha(R - K) + \beta(R^* - K^*)] + (n - 2)[\beta(R - K) + \alpha(R^* - K^*)] + F_b^a \rho_a^b. \end{aligned}$$

From (2.8) we obtain $-\varphi \rho_{rj} + (4 - n)F_r^a \rho_{aj} = h_1 g_{jr} + \varphi f_1 F_{jr}$. From (2.8) and this last equation we easily find that

$$[(4 - n)^2 - \varphi^2]\rho_{rj} = [(4 - n)f_1 + \varphi h_1]g_{jr} + [(4 - n)h_1 + \varphi f_1]F_{jr}.$$

Since $p > 2$ and $g > 2$, $(4 - n)f_1 - \varphi^2 \neq 0$, and we can express the preceding relation in the form

$$\rho_{rj} = \frac{(4 - n)f_1 + \varphi h_1}{(4 - n)^2 - \varphi^2}g_{jr} + \frac{(4 - n)h_1 + \varphi f_1}{(4 - n)^2 - \varphi^2}F_{jr}.$$

This shows that ρ_{rj} has the form (2.1). Therefore, we have

THEOREM 1. *The connection (1.3) satisfies (2.1) iff (2.7) holds.*

The tensor on the right hand side of (2.7) is the product concircular curvature tensor [3]. Because of that we introduce the following

Definition. The connection (1.3) satisfying (2.1) is called *PS-concircular (product semisymmetric concircular) connection*.

Contracting (2.7) with respect to i and j , we find

$$(2.9) \quad \begin{aligned} & R_{rk} + [\varphi(\beta R + \alpha R^*) + (n-2)(\alpha R + \beta R^*)]g_{kr} + \\ & \quad + [(n-2)(\beta R + \alpha R^*) + \varphi(\alpha R + \beta R^*)]F_{kr} = \\ & = K_{rk} + [\varphi(\beta K + \alpha K^*) + (n-2)(\alpha K + \beta K^*)]g_{kr} + \\ & \quad + [(n-2)(\beta K + \alpha K^*) + \varphi(\alpha K + \beta K^*)]F_{kr}. \end{aligned}$$

Conversely, we suppose that for (1.3), we have (2.9). Then, substituting $R_{rk} - K_{rk}$ from (2.9) into (1.9), we obtain (2.8). Therefore, we have

THEOREM 2. (1.3) is a PS-concircular connection iff (2.9) holds.

Using the abbreviation

$$(2.10) \quad A = \alpha R + \beta R^*, \quad B = \beta R + \alpha R^*, \quad P = \alpha K + \beta K^*, \quad Q = \beta K + \alpha K^*,$$

we express (2.9) in the form

$$\begin{aligned} & R_{rk} + [\varphi B + (n-2)A]g_{kr} + [(n-2)B + \varphi A]F_{kr} = \\ & = K_{rk} + [\varphi Q + (n-2)P]g_{kr} + [(n-2)Q + \varphi P]F_{kr} \end{aligned}$$

From this, we easily obtain

$$\begin{aligned} R_{rk}^* & = [\varphi B + (n-2)A]F_{kr} + [(n-2)B + \varphi A]g_{kr} = \\ & = K_{rk}^* + [\varphi + (n-2)P]F_{kr} + [(n-2)Q + \varphi P]g_{kr}. \end{aligned}$$

From these two relations it follows that

$$(2.11) \quad \begin{aligned} & \delta_j^i R_{rk} + [\varphi B + (n-2)A]\delta_j^i g_{kr} + [\varphi A + (n-2)B]\delta_j^i F_{kr} = \\ & = \delta_j^i K_{rk} + [\varphi Q + (n-2)P]\delta_j^i g_{kr} + [\varphi P + (n-2)Q]\delta_j^i F_{kr}, \end{aligned}$$

$$(2.12) \quad \begin{aligned} & F_j^i R_{rk} + [\varphi B + (n-2)A]F_j^i g_{kr} + [\varphi A + (n-2)B]F_j^i F_{kr} = \\ & = F_j^i K_{rk} + [\varphi Q + (n-2)P]F_j^i g_{kr} + [\varphi P + (n-2)Q]F_j^i F_{kr}, \end{aligned}$$

$$(2.13) \quad \begin{aligned} & \delta_j^i R_{rk}^* + [\varphi B + (n-2)A]\delta_j^i F_{kr} + (n-2)B + \varphi A] \delta_j^i g_{kr} = \\ & = \delta_j^i K_{rk}^* + [\varphi Q + (n-2)P]\delta_j^i F_{kr} + [(n-2)Q + \varphi P]\delta_j^i g_{kr}, \end{aligned}$$

$$(2.14) \quad \begin{aligned} & F_j^i R_{rk}^* + [\varphi B + (n-2)A]F_j^i F_{kr} + [(n-2)B + \varphi A]F_j^i g_{kr} = \\ & = F_j^i K_{rk}^* + [\varphi Q + (n-2)P]F_j^i F_{kr} + [(n-2)Q + \varphi P]F_j^i g_{kr}. \end{aligned}$$

We multiply (2.11) with $n-2$ and (2.13) with φ and subtract the second from the first. Then, taking into account the notations (1.15), we have

$$(2.15) \quad \begin{aligned} & A\delta_j^i g_{kr} + B\delta_j^i F_{kr} - P\delta_j^i g_{kr} - Q\delta_j^i F_{kr} = \\ & 2b(\delta_j^i R_{rk} - \delta_j^i K_{rk}) - 2a(R_{rk}^* \delta_j^i - K_{rk}^* \delta_j^i). \end{aligned}$$

Now, we multiply (2.12) with φ and (2.14) with $n-2$ and subtract the first from the second. We find

$$(2.16) \quad \begin{aligned} & BF_j^i g_{kr} + AF_j^i F_{kr} - QF_j^i g_{kr} - PF_j^i F_{kr} = \\ & - 2a(F_j^i R_{rk} - F_j^i K_{rk}) + 2b(F_j^i R_{rk}^* - F_j^i K_{rk}^*). \end{aligned}$$

On the other hand, we can express (2.7), using the notation (2.10), in the form:

$$\begin{aligned} R_{rkj}^i &= K_{rkj}^i + (A\delta_k^i g_{jr} + B\delta_k^i F_{jr}) - (P\delta_k^i g_{jr} + Q\delta_k^i F_{jr}) - \\ &\quad - (A\delta_j^i g_{kr} + B\delta_j^i F_{kr}) + (P\delta_j^i g_{kr} + Q\delta_j^i F_{kr}) + \\ &\quad + (AF_k^i F_{jr} + BF_k^i g_{jr}) - (PF_k^i F_{jr} + QF_k^i g_{jr}) - \\ &\quad - (AF_j^i F_{kr} + BF_j^i g_{kr}) + (PF_j^i F_{kr} + QF_j^i g_{kr}). \end{aligned}$$

Substituting from (2.15) and (2.16), we obtain

$$\begin{aligned} R_{rkj}^i &= -2b(\delta_k^i R_{rj} - \delta_j^i R_{rk} + F_k^i R_{rj}^* - F_j^i R_{rk}^*) + \\ &\quad + 2a(\delta_k^i R_{rj}^* - \delta_j^i R_{rk}^* + F_k^i R_{rj} - F_j^i R_{rk}) = \\ R_{rkj}^i &= -2b(\delta_k^i K_{rj} - \delta_j^i K_{rk} + F_k^i K_{rj}^* - F_j^i K_{rk}^*) + \\ &\quad + 2a(\delta_k^i K_{rj}^* - \delta_j^i K_{rk}^* + F_k^i K_{rj} - F_j^i K_{rk}). \end{aligned}$$

But the tensor on the right-hand side is the product projective curvature tensor [7]. Thus we have

THEOREM 3. *The product projective curvature tensor is an invariant of the PS-concircular connection.*

Lowering the index i in the preceding equation and taking into account (1.11), and then raising the index r , we obtain

$$\begin{aligned} (2.17) \quad R_{ikj}^r &= -2b(g_{ik} R_j^r - g_{ij} R_k^r + F_{ik} R_j^{*r} - F_{ij} R_k^{*r}) + \\ &\quad + 2a(g_{ik} R_j^{*r} - g_{ij} R_k^{*r} + F_{ik} R_j^r - F_{ij} R_k^r) = \\ R_{rkj}^r &= -2b(g_{ik} K_j^r - g_{ij} K_k^r + F_{ik} K_j^{*r} - F_{ij} K_k^{*r}) + \\ &\quad + 2a(g_{ik} K_j^{*r} - g_{ij} K_k^{*r} + F_{ik} K_j^r - F_{ij} K_k^r), \end{aligned}$$

where $R_k^{*r} = R_a^r F_k^a$, $K_k^{*r} = K_a^r F_k^a$.

Conversely, we suppose that for (1.3) we have (2.17). Contracting (2.17) with respect to r and j , we get

$$\begin{aligned} (2.18) \quad & -[n(n-2) - \varphi^2]R_{ik} + 2\varphi R_{ik}^* + [n(n-2) - \varphi^2]K_{ik} - 2\varphi K_{ik}^* = \\ & = -[(n-2)R - \varphi R^*]g_{ik} - [(n-2)R^* - \varphi R]F_{ik} + \\ & + [(n-2)K - \varphi K^*]g_{ik} + [(n-2)K^* - \varphi K]F_{ik}. \end{aligned}$$

From (2.18) it follows that

$$\begin{aligned} (2.19) \quad & 2\varphi R_{ik} - [n(n-2) - \varphi^2]R_{ik}^* - 2\varphi K_{ik} + [n(n-2) - \varphi^2]K_{ik}^* = \\ & = -[(n-2)R^* - \varphi R]g_{ik} - [(n-2)R - \varphi R^*]F_{ik} + \\ & + [(n-2)K^* - \varphi K]g_{ik} + [(n-2)K - \varphi K^*]F_{ik}. \end{aligned}$$

We multiply (2.18) with $n(n-2)-\varphi^2$ and (2.19) with 2φ and add the obtained relations. Then we have

$$\begin{aligned} R_{ik} - K_{ik} = & - \frac{[n-2]R - \varphi R^* [n(n-2) - \varphi^2] + 2\varphi[(n-2)R^* - \varphi R]}{4\varphi^2 - [n(n-2) - \varphi^2]^2} g_{ik} - \\ & - \frac{[n-2]R^* - \varphi R [n(n-2) - \varphi^2] + 2\varphi[(n-2)R - \varphi R^*]}{4\varphi^2 - [n(n-2) - \varphi^2]^2} F_{ik} + \\ & + \frac{[n-2]K - \varphi K^* [n(n-2) - \varphi^2] + 2\varphi[(n-2)K^* - \varphi K]}{4\varphi^2 - [n(n-2) - \varphi^2]^2} g_{ik} - \\ & - \frac{[n-2]K^* - \varphi K [n(n-2) - \varphi^2] + 2\varphi[(n-2)K - \varphi K^*]}{4\varphi^2 - [n(n-2) - \varphi^2]^2} F_{ik}. \end{aligned}$$

(Since $p > 2$, and $q > 2$, $4\varphi^2 - [n(n-2) - \varphi^2]^2 \neq 0$.)

Substituting $R_{rk} - K_{rk}$ from this relation into (1.9), we obtain a relation of the form (2.8). Therefore, we have

THEOREM 4. *The connection (1.3) is a PS-concircular connection iff (2.17) holds.*

3. Product coharmonic curvature tensor. In this section we consider another special product semisymmetric metric F-connection, namely the connection (1.3) satisfying the conditions

$$(3.1) \quad \rho_a^a = 0, \quad \rho_b^a F_a^b = 0.$$

Transvecting (1.9) and (1.10) with g_{rk} , we obtain

$$(3.2) \quad \begin{aligned} R &= K + 2(n-2)\rho_a^a + 2\varphi F_b^a \rho_a^b, \\ R^* &= K^* + 2\varphi \rho_a^a + 2(n-2)F_b^a \rho_a^b. \end{aligned}$$

So, if the connection (1.3) satisfies (3.1), we have $R = K$, $R^* = K^*$ and the relation (1.12) reduces to

$$(3.3) \quad R_{rkj}^i + b_1 s_{rkj}^{-i} - a_1 s_{akj}^{-i} F_r^a = K_{rkj}^i + b_1 s_{rkj}^i - a_1 s_{akj}^i F_r^a.$$

Conversely, we suppose that for (1.3) we have (3.3). Then, contracting (3.3) with respect to i and j , we get

$$\begin{aligned} & [1 + b_1(n-4) - a_1\varphi]R_{rk} + [b_1\varphi - a_1(n-4)]R_{rk}^* + (b_1R - a_1R^*)g_{rk} + \\ & \quad + (b_1R^* - a_1R)F_{rk} = \\ & = [1 + b_1(n-4) - a_1\varphi]K_{rk} + [b_1\varphi - a_1(n-4)]K_{rk}^* + (b_1R - a_1K^*)g_{rk} + \\ & \quad + (b_1K^* - a_1K)F_{rk}. \end{aligned}$$

Transvecting this equation with g^{rk} and F^{rk} , after some calculation, we obtain

$$(3.4) \quad [n(n-4) - \varphi^2]R - 4R^* = [n(n-4) - \varphi]K - 4K^*,$$

$$(3.5) \quad -4R + [n(n-4) - \varphi^2]R^* = -4K + [n(n-4) - \varphi^2]K^*.$$

We multiply (3.4) with $n(n-4) - \varphi^2$ and (3.5) with 4 and add the obtained relations. Then we have

$$\{[(n(n-4) - \varphi^2)^2 - 16]\}R = \{[(n(n-4) - \varphi^2)^2 - 16]\}K.$$

Since $p > 2$ and $q > 2$, $[n(n-4) - \varphi^2]^2 - 16 \neq 0$, and therefore $R = K$. Thus (3.2) reduces to

$$(n-2)\rho_a^a + \varphi F_b^a \rho_a^b = 0, \quad \varphi \rho_a^a + (n-2)F_b^a \rho_a^b = 0,$$

from which $\rho_a^a = 0$ and $F_b^a \rho_a^b = 0$.

Therefore, we have

THEOREM 5. (1.3) satisfies the condition (3.1) iff (3.3) holds.

The tensor on the right-hand side of (3.3) is analogous to the conharmonic curvature tensor [2]. Because of that we introduce the following

Definition. The tensor on the right-hand side of (3.3) is called a *product coharmonic curvature tensor*.

The connection (1.3) satisfying (3.1) is called a PS-coharmonic connection.

4. Holomorphically semisymmetric connections. The geometrical meaning of the semisymmetric connection was given by E. Bartolotti [1] and it consists in the following. Let U and V be two vectors. The vectors $S_{ij}^k u^i v^j$, u^k , v^k are, in the general case, linearly independent. But if

$$(4.1) \quad S_{ij}^k u^i v^j = pu^k + qv^k$$

for every U and V , where p and q are scalars, then S_{ij}^k has the form (0.1), and conversely.

To generalize this property in the case of the locally decomposable Riemannian space, we considered in [5] the skew-symmetric tensor S_{ij}^k satisfying the condition $S_{ij}^k u^i F_a^j u^a = pu^k + qF_a^k u^a$ instead of (4.1) and proved the following:

THEOREM. The skew-symmetric tensor S_{ij}^k satisfies condition (4.1) for every U iff it has the form

$$(4.2) \quad S_{ij}^k = \delta_j^k S_i - \delta_i^k S_j = F_j^k F_i^a S_a + F_i^k F_j^a S_a + (\delta_i^a \delta_j^b + F_i^a F_j^b) w_{ab}^k / 2,$$

where w_{ab}^k is an arbitrary skew-symmetric tensor.

One connection whose torsion tensor has the form (4.2) is the connection

$$(4.3) \quad G_{ij}^k = \left\{ \begin{matrix} k \\ ij \end{matrix} \right\} + \delta_j^k S_i + \varepsilon g_{ij} S^k - F_j^k F_i^a S_a + \varepsilon F_{ij} F_a^k S^a,$$

where $\varepsilon = +1$ or $\varepsilon = -1$. (The case $\varepsilon = -1$ resembles more to the classical semisymmetric metric connection, i.e. to the connection $\left\{ \begin{matrix} k \\ ij \end{matrix} \right\} + \delta_j^k S_i - g_{ij} S^k$, but

the obtained results hold good for the case $\varepsilon = +1$, too.) This connection is an F-connection, i.e. $\partial F/\partial x^k + G_{ka}^i F_j^a - G_{kj}^a F_a^i = 0$, but is not a metric one.

Definition. The connection c(4.3) is called a *holomorphically semisymmetric (HS)-connection*.

The curvature tensor H_{rkj}^i of the connection (4.3) can be expressed in the form [5]

$$H_{rkj}^i = K_{rkj}^i + \delta_r^i \psi_{jk} - \delta_r^i \psi_{kj} + F_r^i F_j^a \psi_{ak} - F_r^i F_k^a \psi_{aj} + \\ + g_{jr} \psi_k^i - g_{kr} \psi_j^i + F_{jr} F_a^i \psi_k^a - F_{kr} F_a^i \psi_j^a,$$

where $\psi_{jk} = \varepsilon \nabla_k S_j + S_j S_k + F_j^a F_k^b S_a S_b$, $\psi_k^i = g^{ia} \psi_{ak}$.

As in 1, we suppose that S_i is locally a gradient and satisfies the condition (1.4). Then

$$(4.4) \quad \psi_{jk} = \psi_{kj}, \quad F_j^a \psi_{ak} = F_k^a \psi_{aj}$$

and the preceding equation reduces to

$$(4.5) \quad H_{rkj} = K_{rkj}^i + g_{jr} \psi_k^i + g_{kr} \psi_j^i + F_{jr} F_a^i \psi_k^a - F_{kr} F_a^i \psi_j^a.$$

If $H_{rkj}^i = H_{rka}^a$, $H_{rk}^* = H_{ak} F_r^a$, $H = H_a^a$, $H^* = H_a^{*a}$, from (4.5) we obtain

$$(4.6) \quad H_{rk} = K_{rk} + 2\psi_{rk} - g_{kr} \psi_a^a - F_{kr} F_a^b \psi_a^b,$$

$$(4.7) \quad H_{rk}^* = K_{rk}^* + 2\psi_{ak} F_r^a F_{kr} \psi_a^a - g_{kr} F_a^b \psi_b^a.$$

Transvecting (4.5) and (4.6) with g^{rk} , we find

$$(4.8) \quad H - K = (2 - n) \psi_a^a - \varphi F_a^b \psi_b^a, \quad H^* - K^* = -\varphi \psi_a^a + (2 - n) F_a^b \psi_b^a.$$

Taking into account (4.6), (4.7) and (4.8), we can eliminate ψ_k^i from (4.5) and we obtain the relation

$$(4.9) \quad \begin{aligned} & H_{rkj}^i - (g_{jr} H_k^i - g_{kr} H_j^i + F_{jr} H_k^{*i} - F_{kr} H_j^{*i})/2 - \\ & - (bH - aH^*) r_{rkj}^i - (bH^* - aH) r_{akj}^i F_r^a = \\ & = K r k i - (g_{jr} K_k^i - g_{kr} K_j^i + F_{jr} K_k^{*i} - F_{kr} K_k^{*i})/2 - \\ & - (bK - QK^*) r_{rkj}^i - (bK^* - aK) r_{akj}^i F_r^a \end{aligned}$$

where we have used the notations (1.14) and (1.15).

5. HS-concircular connection. In this section we consider a special HS-connection, namely the connection (4.3) satisfying the condition

$$(5.1) \quad \psi_k^i = f \delta_k^i + h F_k^i,$$

where f and h are some scalar functions.

Substituting (5.1) into (4.5), we find

$$(5.2) \quad H_{rkj}^i = K_{rkj}^i + fr_{rkj}^i + h_{arj}^i F_r^a.$$

Contracting with respect to i and j , we get

$$H_{rk} = K_{rk} + g_{rk}[(2-n)f - \varphi h] + F_{rk}[(2-n)h - \varphi f].$$

Transvecting this with g^{rk} and F^{rk} , we find

$$(5.3) \quad H - K = [n(2-n) - \varphi^2]f + 2\varphi(1-n)h$$

$$(5.4) \quad H^* - K^* = 2\varphi(1-n)f + [n(2-n) - \varphi^2]h.$$

We multiply (5.3) with $n(2-n) - \varphi^2$ and (5.4) with $2(1-n)\varphi$ and subtract the second from the first. Afterward, we multiply (5.3) with $2(1-n)\varphi$ and (5.4) with $n(2-n) - \varphi^2$ and subtract the first from the second. Then, taking into account the notations (2.6), we have

$$(5.5) \quad f = \alpha(H - K) + \beta(H^* - K^*), \quad h = \beta(H - K) + \alpha(H^* - K^*)$$

Substituting (5.5) into (5.2), we find

$$(5.6) \quad \begin{aligned} H_{rkj}^i - (\alpha H + \beta H^*)r_{rkj}^i - (\beta H + \alpha H^*)r_{akj}^i F_r^a = \\ = K_{rkj}^i - (\alpha K + \beta K^*)r_{rkj}^i - (\beta K + \alpha K^*)r_{arj}^i F_r^a. \end{aligned}$$

Conversely, we suppose that for (4.3) we have (5.6). Then, substituting $H_{rkj}^i - K_{rkj}^i$ from (5.6) into (4.5), we find

$$\begin{aligned} g_{jr}\psi_k^i - g_{kr}\psi_j^i + F_{jr}F_a^i\psi_k^a - F_{kr}F_a^i\psi_j^a = \\ -(\alpha H + \beta H^* - \alpha K - \beta K^*)r_{rkj}^i + (\beta H + \alpha H^* - \beta K - \alpha K^*)r_{akj}^i F_r^a. \end{aligned}$$

Contracting with respect to i and j we obtain

$$(5.7) \quad \begin{aligned} 2\psi_{rk} = \\ g_{rk}\{[\alpha(H - K) + \beta(H^* - K^*)](2-n) - [\beta(H - K) + \alpha(H^* - K^*)]\varphi + \psi_a^a\} + \\ + F_{rk}\{-[\alpha(H - K) + \beta(H^* - K^*)]\varphi + [\beta(H - K) + \alpha(H^* - K^*)](2-n) + F_a^b\psi_b^a\} \end{aligned}$$

and this is an equation of the form (5.1). Therefore, we have

THEOREM 6. *The connection (4.3) satisfies the condition (5.1) iff (5.6) holds.*

The tensor on the right-hand side of (5.6) being the product concircular curvature tensor, it is reasonable to introduce the following

Definition. The connection (4.3) satisfying (5.1) is called a *HS - concircular connection*.

Now, we contract (5.6) with respect to i and j and find

$$\begin{aligned}
 (5.8) \quad & H_{rk} + [(n-2)(\alpha H + \beta H^*) + \varphi(\beta H + \alpha H^*)g_{rk} + \\
 & + [\varphi(\alpha H + \beta H^*) + (n-2)(\beta H + \alpha H^*)F_{rk} = \\
 & = K_{rk} + [(n-2)(\alpha K + \beta K^*) + \varphi(\beta K + \alpha K^*)g_{rk} + \\
 & + [\varphi(\alpha K + \beta K^*) + (n-2)(\beta K + \alpha K^*)]F_{rk}.
 \end{aligned}$$

Conversely we suppose that for the HS-connection (4.3) we have (5.8). Then substituting $H_{rk} - K_{rk}$ from (5.8) into (4.6) we obtain (5.7), i.e. we obtain an equation of the form (5.1). Therefore, we have

THEOREM 7. *An HS-connection is HS-concircular iff (5.8) holds.*

In the same way as in section 2, using the relation (5.8) we obtain

$$\begin{aligned}
 & H_{rkj}^i - 2b(\delta_k^i H_{rj} - \delta_j^i H_{rk} + F_k^i H_{rj}^* - F_j^i H_{rk}^*) + \\
 & + 2a(\delta_k^i H_{rj}^* - \delta_j^i H_{rk}^* + F_k^i H_{rj} - F_j^i H_{rk}) = \\
 & = K_{rkj}^i - 2b(\delta_k^i K_{rj} - \delta_j^i K_{rk} + F_k^i K_{rj}^* - F_j^i K_{rk}^*) + \\
 & + 2a(\delta_k^i K_{rj}^* - \delta_j^i K_{rk}^* + F_k^i K_{rj} - F_j^i K_{rk}),
 \end{aligned}$$

and therefore we have

THEOREM 8. *The product projective curvature tensor is an invariant of the HS-concircular connection.*

6. Another special HS-connection. In this section we consider HS-connections satisfying the conditions

$$(6.1) \quad \psi_a^\alpha = 0, \quad \psi_b^\alpha F_a^b = 0.$$

Then (4.8) gives $H = K$ and $H^* = K^*$ and (4.9) reduces to

$$\begin{aligned}
 (6.2) \quad & H_{rkj}^i - (g_{jr} H_k^i - g_{kr} H_j^i + F_{jr} H_k^{*i} - F_{kr} H_j^{*i})/2 = \\
 & = K_{rkj}^i - (g_{jr} K_k^i - g_{kr} K_j^i + F_{jr} K_k^{*i} - F_{kr} K_j^{*i})/2.
 \end{aligned}$$

Conversely, we suppose that for the HS-connection (4.3) we have (6.2). Then contracting (6.2) with respect to i and j , we find

$$(H - K)g_{rk} + (H^* - K^*)F_{rk} = O.$$

Transvecting this relation with g^{rk} and F^{rk} , we obtain

$$(H - K)n + (H^* - K^*)\varphi = 0, \quad (H - K)\varphi + (H^* - K^*)n = 0.$$

Consequently $H = K$ and $H^* = K^*$ and (4.8) reduces to

$$(2 - n)\psi_a^\alpha - \varphi\psi_b^\alpha F_a^b = 0, \quad -\varphi\psi_a^\alpha + (2 - n)F_a^b \psi_b^\alpha = 0,$$

from which (6.1) follows. Therefore we have

THEOREM 9. *An HS-connection satisfies the condition (6.1) iff (6.1) holds.*

7. Remark concerning HS-connections. Lowering the index i in (4.5) we have

$$H_{irkj} = 2K_{irkj} + g_{jr}\psi_{ik} - g_{kr}\psi_{ij} + F_{jr}F_{ia}\psi_k^a - F_{kr}F_{ia}\psi_j^a.$$

From this we obtain

$$\begin{aligned} H_{irkj} - H_{rikj} &= 2K_{irkj} - g_{ji}\psi_{rk} + g_{ki}\psi_{rj} - F_{ji}F_r^a\psi_{ak} + F_{ki}F_r^a\psi_{aj} + \\ &\quad + g_{jr}\psi_{ik} - g_{kr}\psi_{ij} + F_{jr}F_i^a\psi_{ak} - F_{kr}F_i^a\psi_{aj}. \end{aligned}$$

Let us introduce the following notation

$$L_{irkj} = (H_{irkj} - H_{rikj})/2, \quad \rho_{rk} = -\psi_{rk}/2.$$

Then we express the preceding relation in the form

$$\begin{aligned} L_{irkj} &= K_{irkj} + g_{ji}\rho_{rk} - g_{ki}\rho_{rj} + F_{ji}F_r^a\rho_{ak} - F_{ki}F_r^a\rho_{aj} - \\ &\quad - g_{jr}\rho_{ik} + g_{kr}\rho_{ij} - F_{jr}F_i^a\rho_{ak} + F_{kr}F_i^a\rho_{aj}. \end{aligned}$$

or, raising the index i , in the form

$$\begin{aligned} L_{rkj}^i &= K_{rkj}^i + \delta_j^i\rho_{rk} - \delta_k^i\rho_{rj} + F_j^iF_r^a\rho_{ak} - F_{ki}F_r^a\rho_{aj} - \\ &\quad - g_{jr}\rho_k^i + g_{kr}\rho_j^i - F_{jr}F^{ia}\rho_{ak} + F_{kr}F^{ia}\rho_{aj}. \end{aligned}$$

The right-hand side of this relation has the same form as the right-hand side of (1.6). Therefore, all conclusions of 2 and 3 can be repeated for HS-connections and the tensor L_{rkj}^i .

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