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ABELIAN TYPE THEOREMS FOR SOME INTEGRAL OPERATORS IN $\mathbf{R}^n$

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Abstract. For integral transforms of functions defined on cones in the n-dimensional Euclidean space we prove two theorems of the following "Abelian" type: The transform of a regularly varying function is regularly varying.

1. Introduction. Let, as usual, $\mathbf{R}^n$ denote the real n-dimensional Euclidean space. If $x = (\xi_1, \dots, \xi_n)$ and $y = (\eta_1, \dots, \eta_n)$ are elements of $\mathbf{R}^n$, their inner product is denoted by $x \cdot y = \sum_{j=1}^n \xi_j \eta_j$ and the norm of x by $|x| = (x \cdot x)^{1/2}$.

The set $\Gamma \subset \mathbf{R}^n$ is a *cone* if $x \in \Gamma$ implies $\lambda x \in \Gamma$, for every $\lambda > 0$. An example of cone is $\mathbf{R}^n_+ = \{x \in \mathbf{R}^n : \xi_1 > 0, \dots, \xi_n > 0\}$, the positive octant. We shall always assume that the cone Γ is closed, convex, that it has nonempty interior and that it is acute; see [**3**] or [**4**].

When examining the asymptotic relations the notion of regular variation is very useful. Regularly varying functions were first defined by Karamata – in the one-dimensional case (see [**2**]). This definition was generalized to the n-dimensional case by Yakymiv in the following way.

Definition [**5**]. Let Γ be a cone in $\mathbf{R}^n$. A measurable function $R : \Gamma \to \mathbf{R}_+$ is *regularly varying* (at infinity) if a function $\varphi : \Gamma \to \mathbf{R}_+$ exists such that

$$\lim_{\lambda \to \infty} \sup_{x \in B} \left| \frac{R(\lambda x)}{R(\lambda e)} - \varphi(x) \right| = 0 \tag{1}$$

for a fixed $e \in \Gamma$ $(e \neq 0)$ and for every compact set $B \subset \Gamma \setminus \{0\}$.

Recall that in the definition of regularly varying functions in one variable only the existence of the pointwise limit of $R(\lambda x)/R(\lambda)$ is required and that (the analogue of) (1) follows by the Uniform Convergence Theorem [**2**]. It was proved by Yakymiv that for $n \geq 2$ the existence of the pointwise limit would not imply the existence of the uniform limit and this motivates the Definition.

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It is easily seen that the class of functions satisfying the Definition does not depend on the choice of e; we can assume that $\varphi(e) = 1$.

The function φ in (1) is homogeneous, i.e. there is an $\rho \in \mathbf{R}$, called the order of φ, such that $\varphi(\lambda x) = \lambda^\rho \varphi(x)$, for every $\lambda > 0$ and $x \in \Gamma$; moreover, φ is continuous and bounded away from zero. The order of φ is also called the index of the regularly varying function R. We shall consistently use the following notation: R will stand for a regularly varying function, φ for its "index" function, r for the function in one variable $R(\lambda e)$ (which is obviously regularly varying in $\mathbf{R}_+$).

A regularly varying function whose index function is 1 is called a *slowly varying function*. Every regularly varying function R can be represented in the following way $R = \varphi L$, where φ is the index function and L is a slowly varying function.

Finally, let us define regular variation at zero. A measurable function $R : \Gamma \to \mathbf{R}_+$ is *regularly varying at zero* (with index ρ) if the limit

$$\lim_{\lambda\to\infty} \frac{R(x/\lambda)}{r(\lambda)} = \varphi(x)$$

exists uniformly in $x \in B$, for every compact set $B \subset \Gamma \setminus \{0\}$.

The main objective of the present paper is the investigation of some integral operators of the form

$$\mathcal{K}F(x) = \int_\Gamma F(u)k(u,x)du, \qquad x \in G, \tag{2}$$

which transform functions defined on a cone Γ into functions defined on a cone G. We shall consider two kinds of such operators (the first applicable to arbitrary regularly varying functions – Section 2, and the second only to the monotone ones – Section 3) and we shall prove that for these operators the regular variation of F implies the regular variation of $\mathcal{K}F$. In Section 4 we deduce analogous results for the regular variation at zero. These statements are n-dimensional counterparts of the one-dimensional results from [**1**].

Let us first state some facts about regularly varying functions.

(i) A regularly varying function on Γ is locally bounded on Γ (= bounded on every compact set $B \subset \Gamma \setminus \{0\}$) [**5**].

(ii) For every slowly varying function L there is an asymptotically equivalent radial slowly varying function L_r [**5**].

This is easily seen by putting $L_r(x) = L(|x|e)$, for some $e \in \Gamma$; let l be defined by $l(|x|) = L_r(x)$.

Let L be a slowly varying function on Γ and let F be a locally integrable function on Γ. Let $|u| < \delta$ denote the set $\{u \in \Gamma : |u| < \delta\}$ and similarly for

$|u| > \Delta$. Then for some $\eta > 0$ and some positive constants C_1 and C_2 we have

$$\text{(iii)} \qquad \int\limits_{|u|<\delta} L(\lambda u)F(u)du \leq C_1\delta^\eta l(\lambda\delta) \int\limits_{|u|<\delta} |u|^{-\eta}F(u)du$$

$$\text{(iv)} \qquad \int\limits_{|u|<\delta} L(\lambda u)F(u)du \leq C_2\Delta^{-\eta} l(\lambda\Delta) \int\limits_{|u|>\Delta} |u|^{\eta}F(u)du$$

for λ large enough.

The proof of (iii) and (iv) is similar to the proof of the corresponding one-dimensional statements [**2**]. For (iii) we use the following property of slowly varying functions $\sup\limits_{|u|<|x|} |u|^\eta L(u) \sim |x|^\eta l(|x|)$, for $\eta > 0$, $|x| \to \infty$ (see [**5**]), and similarly for (iv).

2. Operators with absolutely integrable kernels. Let Γ and G be two cones in $\mathbf{R}^n$. A measurable function $k : \Gamma \times G \to \mathbf{R}$ will be called a *kernel* (on $\Gamma \times G$). In this section we shall consider kernels which satisfy the following condition. For a given $\rho \in \mathbf{R}$

$$\text{(A)} \qquad \int\limits_{\Gamma} \max(|u|^{-\eta+\rho}, |u|^{\eta+\rho})|k(u,x)|du \leq C(x), \quad x \in G,$$

for some $\eta > 0$ and a positive function $C(x)$.

The number ρ is called the *index* of the kernel k.

Now consider an operator defined as in (2). Assume moreover that the function k is homogeneous of order α, for some $\alpha \in \mathbf{R}$ (i.e. $k(\lambda u, \lambda x) = \lambda^\alpha k(u,x)$). Then we have

$$\text{(3)} \qquad \mathcal{K}F(\lambda x) = \lambda^{\alpha+n} \int\limits_{\Gamma} F(\lambda u)k(u,x)du.$$

Indeed, by making a change of variables we have

$$\begin{aligned}\mathcal{K}F(\lambda x) = \int\limits_{\Gamma} F(u)k(u,\lambda x)du &= \lambda^n \int\limits_{\Gamma} F(\lambda\ v)k(\lambda v, \lambda x)dv \\ &= \lambda^{\alpha+n} \int\limits_{\Gamma} F(\lambda v)k(v,x)dv\end{aligned}$$

which proves (3),

Proposition 1. *Let Γ and G be two cones in $\mathbf{R}^n$. Let k be a kernel on $\Gamma \times G$ satisfying (A) with index 0, homogeneous of order α, and let L be a slowly varying function on Γ. Then*

$$\left| \frac{\mathcal{K}L(\lambda x)}{\lambda^{\alpha+n} l(\lambda)} - \mathcal{K}1(x) \right| \leq \varepsilon(\lambda)C(x), \quad x \in G,$$

where $\varepsilon(\lambda) \to 0$, *as* $\lambda \to \infty$.

Proof. We have by (3)

$$\frac{\mathcal{K}L(\lambda x)}{\lambda^{\alpha+n}l(\lambda)} - \mathcal{K}1(x) = \int_\Gamma \left(\frac{L(\lambda u)}{l(\lambda)} - 1\right) k(u,x)du = \tag{4}$$
$$= \int_{|u|<\delta} + \int_{\delta\le|u|\le\Delta} + \int_{\delta\le|u|} = I_1 + I_2 + I_3$$

where δ and Δ will be chosen later.

For the integral I_1 we have

$$|I_1| \le \frac{1}{l(\lambda)} \int_{|u|<\delta} L(\lambda u)|k(u,x)|du + \int_{|u|<\delta} |k(u,x)|du.$$

If we apply property (iii) of slowly varying functions and then condition (A) we obtain

$$|I_1| \le \frac{1}{l(\lambda)} C_1\delta^\eta l(\lambda\delta) \int_{|u|<\delta} |u|^{-\eta}|k(u,x)|du + \int_{|u|<\delta} |k(u,x)|du \le \tag{5}$$
$$\le C_1(l(\lambda\delta)/l(\lambda)+1)\delta^n C(x) \le C_3\delta^\eta C(x)$$

for λ large enough.

Analogously, using (iv) instead of (iii) we have for I_3

$$|I_3| \le C_4\Delta^{-\eta}C(x) \tag{6}$$

for λ large enough.

Now for a given $\varepsilon > 0$ we choose δ and Δ such that $C_3\delta^\eta < \varepsilon$ and $C_4\Delta^{-\eta} < \varepsilon$. Then from (5) and (6) we have

$$|I_1| + |I_3| < \varepsilon C(x) \tag{7}$$

Next for δ and Δ chosen as above consider I_2. Since the set $\delta \le |u| \le \Delta$ is compact it follows from the definition of slowly varying functions that

$$|I_2| < \varepsilon C(x) \tag{8}$$

for λ large enough. Now the proof of the proposition follows by substituting (7) and (8) into (4).

THEOREM 1. *Let* Γ *and* G *be two cones in* $\mathbf{R}^n$. *Let* k *be a kernel satisfying* (*A*) *with index* ρ, *homogeneous of order* α, *and let* R *be u regularly varying function of index* p *in* Γ. *Then*

$$\left|\frac{\mathcal{K}R(\lambda x)}{\lambda^{\alpha+n}r(\lambda)} - \mathcal{K}\varphi(x)\right| \le C(x)\varepsilon(\lambda), \quad x \in G \tag{9}$$

where $\varepsilon(\lambda) \to 0$ *as* $\lambda \to \infty$.

Proof. The function φ is homogeneous of order ρ and continuous and bounded away from zero; therefore

$$C_1|u|^\rho \le \varphi(u) = |u|^\rho \varphi(u') \le C_2|u|^\rho$$

and from this it follows that (A) is equivalent to

$$\int_\Gamma \max(|u|^{-\eta}, |u|^\eta)\varphi(u)k(u,x)du \le C_3 C(x)$$

which means that the function $k_1(u,x) = \varphi(u)k(u,x)$ is a kernel satisfying (A) with index 0. It is moreover homogeneous of order $\rho + \alpha$, and thus it satisfies the conditions of Proposition 1 (with α replaced by $\rho + \alpha$). We shall apply this proposition to the operator $\mathcal{K}_1$ with kernel k_1 and to the slowly varying function $L = R/\varphi$. First observe that

$$\frac{R(\lambda v)}{r(\lambda)} = \frac{L(\lambda v)\varphi(\lambda v)}{l(\lambda)\varphi(\lambda e)} = \frac{L(\lambda v)\lambda^\rho \varphi(v)}{l(\lambda)\lambda^\rho}.$$

From this, by an application of (3), it follows that

$$\frac{\mathcal{K}R(\lambda x)}{\lambda^{\alpha+n} r(\lambda)} - \mathcal{K}\varphi(x) = \int_\Gamma \left(\frac{R(\lambda v)}{r(\lambda)} - \varphi(v)\right) k(v,x)dv =$$
$$= \int_\Gamma \left(\frac{L(\lambda v)}{l(\lambda)} - 1\right) \varphi(v)k(v,x)dv = \frac{\mathcal{K}_1 L(\lambda x)}{\lambda^{\alpha+n+\rho} l(\lambda)} - \mathcal{K}_1 1(x).$$

Now an application of Proposition 1 completes the proof of the theorem.

Remark. We have not proved yet that $\mathcal{K}R$ is regularly varying, for (9) gives only the pointwise convergence of $\mathcal{K}R(\lambda x)/\lambda^{\alpha+n}r(\lambda)$ (and not the uniform). However, it is immediate from (9) that we shall indeed have regular variation if $C(x)$ is assumed to be locally bounded (for instance, $C(x) = C$). We shall formulate this remark as a corollary and observe that similar considerations are in order whenever we have an equation similar to (9) (see (19) in Theorem 2 below).

Corollary 1. *Let k and R satisfy the conditions of Theorem 1 and let the function $C(x)$ in (A) be bounded on compact sets in $G_1 \setminus \{0\}$, where G_1 is a cone contained in, or equal to G. Then $|\mathcal{K}R|$ is regularly varying in G_1 with index function $|\mathcal{K}\varphi|$ and index $\alpha + n + \rho$.*

In the next corollary we shall consider several examples of operators to which Theorem 1 applies.

Let us write $\langle a, b\rangle$ for the *"interval"*, defined by $\langle a, b\rangle = \{x \in \Gamma : a \le_\Gamma x \le_\Gamma b\} = (a + \Gamma) \cap (b - \Gamma)$. We shall write $\rangle a, b\rangle$ for the difference of two intervals $\rangle a, b\rangle = \langle 0, b\rangle \setminus \langle 0, a\rangle$, and also by abusing the notation $\langle a, \infty\rangle = a + \Gamma$ and $\rangle a, \infty\rangle = \Gamma \setminus \langle 0, a\rangle$.

Consider the following three operators:

$$\mathcal{I}_\Gamma f(x) = \int\limits_{\langle o,x\rangle} f(u)du, \quad x \in \Gamma \tag{10}$$

$$\mathcal{J}_\Gamma f(x) = \int\limits_{\langle x,\infty\rangle} f(u)du, \quad x \in \Gamma \tag{11}$$

$$\mathcal{J}_\Gamma^1 f(x) = \int\limits_{\rangle x,\infty\rangle} f(u)du, \quad x \in \Gamma. \tag{12}$$

Corollary 2. *Let R be a regularly varying function on a cone Γ, with index ρ and index function φ.*

a) *Let $-n < \rho < 0$. Then $\mathcal{I}_\Gamma R$ is regularly varying in Γ with index $\rho + n$ and index function $\mathcal{I}_\Gamma \varphi$.*

b) *Let $\rho > -n$. Then $\mathcal{J}_\Gamma R$ and $\mathcal{J}_\Gamma^1 R$ are regularly varying in Γ with index $\rho + n$ and index functions $\mathcal{J}_\Gamma \varphi$ and $\mathcal{J}_\Gamma^1 \varphi$ respectively.*

Proof. We shall prove a) only, the proof of b) being similar. Obviously (10) is an operator of the form (2) with kernel $k(u,x) = \theta_{\langle 0,x\rangle}(u)$ (where θ_A is the characteristic function of the set A). We have to prove that this kernel satisfies the conditions of Corollary 1. Since $k(\lambda u, \lambda x) = k(u,x)$, we see that k is homogeneous of order 0. Next we prove that (A) holds. We have

$$\int\limits_\Gamma \max(|u|^{\rho-\eta}, |u|^{\rho+\eta})|k(u,x)|du = \int\limits_{|u|\le 1} + \int\limits_{|u|>1} = I_1 + I_2.$$

For the first integral we have

$$I_1 = \int\limits_{|u|\le 1} |u|^{\rho-\eta}|k(u,x)|du \le \int\limits_{|u|\le 1} |u|^{\rho-\eta}du \le C \tag{13}$$

if $0 < \eta < \rho + n$; and for the second, if $0 < \eta < -\rho$

$$I_2 = \int\limits_{|u|>1} |u|^{\rho+\eta}|k(u,x)|du \le \int\limits_{|u|>1} k(u,x)du \le \int\limits_{\langle 0,x\rangle} du \equiv V(x). \tag{14}$$

From (13) and (14) it follows that if we choose $\eta < \min(-\rho, \rho+n)$, then $I_1 + I_2 \le C + V(x)$, and this proves (A) with $C(x) = C + V(x)$.

If B is a compact set in $\Gamma \setminus \{0\}$, then $\sup\limits_{x\in B} V(x)$ is bounded, since $V(x)$ is the volume of a bounded set. Now Corollary 1 can be applied to complete the proof.

The functions defined in (10), (11) and (12) may be called primitive functions for f. They are monotone in the following sense.

A cone Γ defines a partial order in $\mathbf{R}^n$. We say that $x \leq_\Gamma y$ if $y - x \in \Gamma$. A real function F is said to be monotone increasing (decreasing) on Γ if $x \leq_\Gamma y$ implies $F(x) \leq F(y)$ $(F(x) \geq F(y))$.

Now, it is obvious that for positive f the function $\mathcal{I}_\Gamma f$ is monotone increasing in Γ, and $\mathcal{J}_\Gamma f$ and $\mathcal{J}_\Gamma^1 f$ are monotone decreasing.

Remark. For monotone functions it is possible to obtain the converse of Corollary 2 (the Tauberian theorem for operators (10), (11) and (12)), i.e., that for monotone R the regular variation of $\mathcal{I}_\Gamma R$ (or $\mathcal{J}_\Gamma R$ or $\mathcal{J}_\Gamma^1 R$) implies the regular variation of R. This was proved (implicitly) by Yakymiv [**5**] (see the proof of Theorem 9.1).

3. Operators with nonabsolutely integrable kernels. In this section we shall consider operators $\mathcal{K}$ with a different kind of kernel (satisfying condition (B) below). These operators will be applied to monotone functions and the integrals defining them will not be absolutely convergent.

Let F be a locally integrable function on Γ. The integral $\int_\Gamma F(t)dt$ is said to converge in the sense of principal value if the limit $\lim_{|b|\to\infty} \int_{\langle 0,b\rangle} F(t)dt$ exists for $b \in \Gamma_1$, where Γ_1 is a cone such that $\Gamma_1 \subset \int \Gamma$.

Let Γ and G be two cones in $\mathbf{R}^n$. In this section we shall consider kernels on $\Gamma \times G$ which satisfy the following condition

$$\text{(B)} \qquad \left| \int_{\langle a,b\rangle} k(u,x)du \right| \leq C(x), \quad x \in G$$

for some positive function $C(x)$ and all intervals $\langle a,b\rangle \subset \Gamma$.

Let us call, for short, the function F *monotone primitive* if there is a positive integrable function f on Γ such that $F(x) = \int_{x+\Gamma} f(t)dt = \mathcal{J}_\Gamma f(x)$.

In the following proposition we prove the existence of the integral $\mathcal{K}F(x)$. This proposition is the analogue of Dirichlet s criterion for nonabsolutely convergent integrals.

PROPOSITION 2. *Let Γ and G be two cones in $\mathbf{R}^n$. Let k be a kernel on $\Gamma \times G$ satisfying (B) and let F be a monotone primitive function on Γ. Then the integral*

$$\mathcal{K}F(x) = \int_\Gamma F(u)k(u,x)du, \quad x \in G \tag{13}$$

exists in the sense of principal value.

The proof of the proposition is based on the following lemma.

Lemma 1. *Let F and k satisfy the conditions of Proposition 2. Let* $F(u) = \int_{u+\Gamma} f(t)dt$. *Then*

$$|I_{a,b}| \equiv \left| \int_{\rangle a,b\rangle} F(u)k(u,x)du \right| \le C(x) \int_{\rangle a,\infty\rangle} f(t)dt. \tag{14}$$

With this lemma, Proposition 2 follows at once. Indeed, to show that (13) converges in the sense of principal value we have to show that the "remainder" $I_{a,b}$ of this integral tends to 0, as $|a|, |b| \to \infty$. Now, by (14) we have $|I_{a,b}| \le C(x) \int_{\rangle a,\infty\rangle} f(t)dt$, and the last integral tends to 0 as $|a| \to \infty$, since f is integrable. This proves Proposition 2.

Proof of Lemma 1. First we shall prove that for k satisfying (B) we have

$$\left| \int_{\rangle a,b\rangle} k(u,x)du \right| \le 2C(x) \tag{15}$$

Indeed, since $\rangle a,b\rangle = \langle 0,b\rangle \setminus \langle 0,a\rangle$ we have $\int_{\rangle a,b\rangle} = \int_{\langle 0,b\rangle} - \int_{\langle 0,a\rangle}$. Now an application of (B) yields (15).

To prove (14) we shall apply Fubini's Theorem to the integral

$$I_{a,b} = \int_{\rangle a,b\rangle} F(u)k(u,x)du = \int_{\rangle a,b\rangle} k(u,x) \int_{u+\Gamma} f(t)dtdu. \tag{16}$$

The domain of integration in the double integral is defined by $u \in \rangle a,b\rangle$, which means (i) $u \in \rangle a,\infty\rangle$ and (ii) $u <_\Gamma b$, and by $t \in u+\Gamma$, which means (iii) $t >_\Gamma u$.

Now, it is easily seen that by (i) and (iii) we have $t \in \rangle a,\infty\rangle$, and that for u we have, by (ii) and (iii), $u <_\Gamma b$ and $u <_\Gamma t$, which, if we put $c = \min_\Gamma(b,t)$, is equivalent with $u <_\Gamma c$; and this together with (i) gives $u \in \rangle a,c\rangle$.

Thus if we reverse the order of integration in (16) we have

$$I_{a,b} = \int_{\rangle a,b\rangle} k(u,x) \int_{u+\Gamma} f(t)dtdu. = \int_{\rangle a,\infty\rangle} f(t) \int_{\rangle a,c\rangle} k(u,x)dudt.$$

And now we apply (15) to the last integral:

$$|I_{a,b}| = \int_{\rangle a,\infty\rangle} f(t) \left| \int_{\rangle a,c\rangle} k(u,x)du \right| dt \le C(x) \int_{\rangle a,\infty\rangle} f(t)\,dt$$

which completes the proof of the lemma.

Now we shall consider the regular variation of monotone primitive functions. First we prove a lemma.

LEMMA 2. *Let R be monotone primitive on a cone Γ, $R(x) = \int_{x+\Gamma} f(t)dt$ such that f is also monotone, and let R be regularly varying with index ρ, $-n < \rho < 0$. Then*

a) *the function f is regularly varying with index function ψ such that $\varphi(x) = \int_{x+\Gamma} \psi(u)du$.*

b) *Let k satisfy condition (B), then*

$$\frac{1}{r(\lambda)}\left|\int_{\rangle a,\infty\rangle} R(\lambda u)k(u,x)du\right| \le C(x) \int_{\rangle a,\infty\rangle} \psi(u)du. \tag{17}$$

Proof. Part a) follows by the remark at the end of Section 2.

b) By Lemma 1 we have

$$\frac{1}{r(\lambda)}\left|\int_{\rangle a,b\rangle} R(\lambda u)k(u,x)du\right| \le \frac{C(x)}{r(\lambda} \int_{\rangle \lambda a,\infty\rangle} f(u)du. \tag{18}$$

But since f is regularly varying, an application of Corollary 2 b) yields that $\mathcal{J}_\Gamma^1 f(a) = \int_{\rangle a,\infty\rangle} f(u)du$ is regularly varying and has the index function $\mathcal{J}_\Gamma^1\psi(a)$. Thus letting $\lambda \to \infty$ in (18) we shall have $1/r(\lambda) \int_{\rangle \lambda a,\infty\rangle} f(u)du \to \int_{\rangle a,\infty\rangle} \psi(u)du$ and this proves (17).

THEOREM 2. *Let Γ and G be two cones in $\mathbf{R}^n$. Let $-n < \rho < 0$. Let R be regularly varying on Γ with index ρ, and let R be monotone primitive, $R(x) = \int_{x+\Gamma} f(t)dt$, such that f is also monotone. Let k be a bounded kernel on $\Gamma \times G$ satisfying (B), homogeneous of order α. Then*

$$\left|\frac{\mathcal{K}R(\lambda x)}{\lambda^{\alpha+n}r(\lambda)} - \mathcal{K}\varphi(x)\right| \le \varepsilon(\lambda)C(x), \quad x \in G \tag{19}$$

where $\varepsilon(\lambda) \to 0$ as $\lambda \to \infty$.

Proof. We have, as in the proof of Theorem 1 (see (3)),

$$\begin{aligned} \frac{\mathcal{K}R(\lambda x)}{\lambda^{\alpha+n}r(\lambda)} - \mathcal{K}\varphi(x) &= \int_\Gamma \left(\frac{R(\lambda v)}{r(\lambda)} - \varphi(v)\right) k(v,x)dv = \\ &= \int_{\rangle 0,a\rangle} + \int_{\rangle a,A\rangle} + \int_{\rangle A,\infty\rangle} = I_1 + I_2 + I_3, \end{aligned} \tag{20}$$

say. Consider I_1. Since R is regularly varying with index ρ, $-n < \rho < 0$, from Corollary 2 a) it follows that $\mathcal{I}_\Gamma R(x) = \int_{\langle 0,x\rangle} R(u)du$ is regularly varying with index function $\mathcal{I}_\Gamma\varphi(x) = \int\limits_{\langle 0,x\rangle} \varphi(u)du$ (and with index $\rho+n$). From this we shall have, since k is bounded

$$|I_1| \le \frac{1}{r(\lambda)} \int\limits_{\langle 0,a\rangle} R(\lambda v)dv + \int\limits_{\langle 0,a\rangle} \varphi(v)dv = \frac{\mathcal{I}_\Gamma R(\lambda a)}{\lambda^n r(\lambda)} + \mathcal{I}_\Gamma\varphi(a) < 2\mathcal{I}_\Gamma\varphi(a)$$

for λ large enough. Now given ε (depending on λ) we can choose a small enough in Γ such that

$$|I_1| < \varepsilon. \tag{21}$$

For the integral I_3 we have

$$|I_3| \le \frac{1}{r(\lambda)}\Bigg|\int\limits_{\rangle A,\infty\rangle} R(\lambda v)k(v,x)dv\Bigg| + \Bigg|\int\limits_{\rangle A,\infty\rangle} \varphi(v)k(v,x)dv\Bigg|$$

and since R satisfies the conditions of Lemma 2, we have by (17)

$$|I_3| \le 2C(x)\int\limits_{\rangle A,\infty\rangle} \psi(u)du.$$

Now if we choose A large enough in Γ we shall have $\int\limits_{\rangle A,\infty\rangle} \psi(u)du < \varepsilon$. Thus we have proved

$$|I_3| < \varepsilon C(x). \tag{22}$$

For a and A chosen as above we consider I_2

$$|I_2| \le \int\limits_{\rangle A,\infty\rangle} \left|\frac{R(\lambda v)}{r(\lambda)} - \varphi(v)\right| dv \to 0 \tag{23}$$

as $\lambda \to \infty$, by the definition of regularly varying functions, since $\rangle a, A\rangle$ is compact in $\Gamma\setminus\{0\}$. By substituting (21), (22) and (23) into (20) we complete the proof of the theorem.

Remark. Obviously, we can obtain as a corollary of this theorem a statement similar to Corollary 1 (see also the Remark after Theorem 1); namely, assuming the local boundedness of $C(x)$ the regular variation of R implies the regular variation of $|\mathcal{K}R|$.

4. Asymptotic behavior at zero. In the present section we prove two corollaries of Theorems 1 and 2.

COROLLARY 4. *Let R and k satisfy the assumptions of Theorem 1, or the assumptions of Theorem 2. Then*

$$\lim_{\lambda\to\infty} \frac{R(u/\lambda)}{r(\lambda)} = \varphi(u), \quad u \in \Gamma,$$

uniformly in compact sets in $\Gamma \setminus \{0\}$*, implies*

$$\lim_{\lambda\to\infty} \frac{\mathcal{K}R(x/\lambda)}{\lambda^{-\alpha-n} r(\lambda)} = \mathcal{K}\varphi(x), \quad x \in G, \tag{23}$$

and the convergence in (23) is uniform in compact sets in $G \setminus \{0\}$*, under the additional assumption that* $C(x)$ *is locally bounded.*

Proof. We have

$$\mathcal{K}R(x/\lambda) = \int_\Gamma R(u)k(u, x/\lambda)du = \int_\Gamma R(v/\lambda)k(v/\lambda, x/\lambda)\lambda^{-n}dv = $$
$$= \lambda^{-n-\alpha} \int_\Gamma R(v/\lambda)k(v,x)dv$$

and if we use this equation instead of (3), the proof of the corollary follows along the same lines as the proof of Theorem 1 or the proof of Theorem 2.

Next we consider kernels which instead of bein homogeneous of order a satisfy the following condition

$$k(\lambda u, x/\lambda) = k(u,x). \tag{24}$$

Corollary 5. *Let* R *and* k *satisfy the assumptions of Theorem 1 or the assumptions of Theorem 2, – only let* k *satisfy (24) instead of being homogeneous of order* α*. Then*

$$\lim_{\lambda\to\infty} \frac{R(\lambda u)}{r(\lambda)} = \varphi(u), \quad u \in \Gamma,$$

uniformly in compact sets in $\Gamma \setminus \{0\}$*, implies*

$$\lim_{\lambda\to\infty} \frac{\mathcal{K}R(x/\lambda)}{\lambda^n r(\lambda)} = \mathcal{K}\varphi(x), \quad x \in G. \tag{25}$$

Here also we have uniform convergence in (25) if the function $C(x)$ is locally bounded and the proof is very similar to the proof of Corollary 4.

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