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NOTE ON SOME MERCERIAN THEOREM

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Abstract. We first give a different proof of a Mercerian theorem in [**3**] for slowly varying sequences, which does not use monotonicity conditions (5). Then we treat the case of regularly varying sequences.

In this paper all the matrices considered *are triangular* and *invertible.* To shorten exposition we shall use the terminology of the first two pages of the paper [**1**] of S. Zimering. In addition, if $A = (a_{nk})$ and s is the sequence with general term s_n, by As we denote the sequence $(As)_n = \sum a_{nk}s_k$. $\Big(\sum$ stands for $\sum_{k=1}^{n}$ unless stated otherwise$\Big)$. If $A_n = \sum a_{nk} = 1$ for all n, we say that the matrix A is *normalized.*

(R_0 denotes the class of *slowly* varying sequences).

We shall use the following two theorems.

THEOREM MV. (M. Vuilleumier [**2**, Th. 4.1] and [**1**, p. 72]).

1° *For a matrix* (a_{nk}) *to be* $O(R_0)$ *-regular, it is necessary and sufficient that, for some* $\gamma > 0$.

$$\sum |a_{nk}|k^{-\gamma} = O(n^{-\gamma}), \qquad (n \to \infty). \tag{1}$$

2° *If in addition to* (1),

$$A_n = \sum a_{nk} \to 1, \qquad (n \to \infty), \tag{2}$$

then the matrix (a_{nk}) *is* R_0*-regular.*

THEOREM SZ. (S. Zimering [**1**, th. A]). *A matrix* (δ_{nk}) *which satisfies the condition*

$$\lim_{n\to\infty} \inf \left\{ |\delta_{nk}| - \sum_{k=1}^{n-1} |\delta_{nk}| \right\} > 0 \tag{3}$$

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is $O(1)$-mercerian. If, in addition, the matrix (δ_{nk}) is regular, then it is mercerian.

In [**3**] N. Tanović-Miller proved the following

Theorem NTM. *Let the matrix $A = (a_{nk})$ be normalized and nonnegative (i.e. $a_{nk} \geq 0$), let, moreover*

$$a_{n1} > 0, \qquad \textit{for all } n, \tag{4}$$

and

$$a_{ni}a_{n-1,k} \leq a_{n-1,i}a_{nk}, \qquad 1 \leq k \leq i \leq n, \quad n \geq 2. \tag{5}$$

If A satisfies the condition (1) *for some $\gamma > 0$, then the matrix*

$$B = (I + \lambda A)/(1 + \lambda), \quad \lambda > -1,$$

is R_0-mercerian, i.e. $B_s \in R_0 \Rightarrow s_n \sim (BS)_n$, $(n \to \infty)$.

(I is the unit matrix. Here we take $B = (I+\lambda A)/(1+\lambda)$ so that the regularity condition is satisfied, i.e. $B = \dfrac{I + \lambda A}{1 + \lambda} \Rightarrow \sum b_{nk} = \dfrac{1 + \lambda \sum a_{nk}}{1 + \lambda} = \dfrac{1 + \lambda}{1 + \lambda} = 1.$)

The proof of Theorem NTM is very elementary in case $\lambda \geq$). However, in case $-1 < \lambda < 0$ the proof becomes somewhat involved, demanding some convexity and inductive arguments.

The aim of this note is to give a different proof of this case (more generally, for $|\lambda| < 1$) which, moreover, does not use monotonicity conditions (5).

Theorem 1. *Let the matrix $A = (a_{nk})$ be normalized and nonnegative. If (4) holds and A satisfies* (1) *for some $\gamma > 0$, then*

$$B = (I + \lambda A)/(1 + \lambda), \qquad |\lambda| < 1,$$

is R_0-mercerian.

Proof. Suppose $Bs \in R_0$. By a well-known propertyh of slowly varying sequences this means: there are sequences $\{c_n\}$ and $\{\varepsilon_n\}$ such that $c_n \to 1$, $\varepsilon \to 0$ $(n \to \infty)$ and $(Bs)_n = c_n\alpha_n$, where $\alpha_n = \exp(\sum \varepsilon_k/k) > 0$. (See [**2**, p. 15], [**4**, p. 45] and [**5**, p. 58–59]).

Thus, we have

$$s_n + \lambda \sum a_{nk}s_k = c_n\alpha_n(1 + \lambda),$$

and, with $t_n = s_n/\alpha_n$,

$$t_n + \lambda \sum a_{nk}\frac{\alpha_k}{\alpha_n}t_k = c_n(1 + \lambda) \to 1 + \lambda \quad (n \to \infty). \tag{6}$$

an Consider the matrix $A' = (a_{nk}\alpha_k/\alpha_n)$. Since $(\alpha_k) \in R_0\}$, we have

$$suma_{nk}\alpha_k \sim \alpha_n, \ \ (n \to \infty), \quad (\sum a_{nk} = 1)$$

i.e.

$$\sum a_{nk}\alpha_k/\alpha_n \to 1, \quad (n \to \infty).$$

(See the last relation in [**2**, p. 19]).

Using now Theorem MV this proves that the matrix A' is R_0-regular, since A' satisfies the same condition (1) as $A(0 < \alpha_k/\alpha_n \leq 1, a_{nk} \geq 0)$. A fortiori, A' is regular.

Let $B' = (I + \lambda A')/(1 + \lambda)$, $|\lambda| < 1$. Trivially, B' is regular. If $B' = (b'_{nk})$, we have

$$(1+\lambda)\left\{|b'_{nn}| - \sum_{k=1}^{n-1} |b'_{nk}|\right\} = |1 + \lambda a_{nn}| - |\lambda| \sum_{k=1}^{n-1} a_{nk}\frac{\alpha_k}{\alpha_N} \geq$$

$$\geq 1 - |\lambda| \sum_{k=1}^{n} a_{nk} = 1 - |\lambda| > 0.$$

Thus, B' satisfies (3) and by Theorem SZ it is mercerian, being regular. This gives $t_n \to 1$, $n \to \infty$, i.e. $s_n = t_n\alpha_n$ where $t_n \to 1$, $(n \to \infty)$. This proves that B is R_0-mercerian.

Let us remark that in a similar way one can treat the case of regularly varying sequences. Namely, in case $\lambda \geq 0$ one applies the method of [**3**] and in case $|\lambda| < 1$ the method of our proof of Theorem 1. However, one can deduce such a theorem as a simple consequence of the theorem NTM [**3**].

(R_σ denotes the class of regularly varying sequences of index σ).

Thus we shall prove

THEOREM 2. *Let the matrix $A = (a_{nk})$ be triangular, normalized and non-negative. Let $a_{n1} > 0$ forall n and $a_{ni}a_{n-1,k} \leq a_{n-1,i}a_{nk}$, $1 \leq k \leq i \leq n$. Let $B = I + \lambda A$ and $\sigma > 0$.*

If A satisfies the conditions $\sum a_{nk}(k/n)^\alpha \to A_\alpha$, $(n \to \infty)$ for some α, $O < \alpha < \sigma$, and for $\alpha = \sigma$, then from $B_s \in R_\sigma$, it follows that

$$s_n \sim \frac{(Bs)n}{1 + \lambda A_\sigma}, \quad (n \to \infty),$$

provided $1 + \lambda A_\sigma > 0$.

Proof. By the well-known representation of regularly varying sequences by slowly varying sequences, from $B_s \in R_\sigma$ follows $((Bs)_n/n^\sigma) \in R_0$. Writing

$$A_\sigma = \lim_{n\to\infty} \sum a_{nk}(k/n)^\sigma$$

we have

$$(Bs)_n/n^\sigma = t_n + \lambda A_\sigma \sum a'_{kn}t_k = (B't)_n$$

where $t_n = s_n/n^\sigma$, $a'_{nk} = a_{nk}(k/n)^\sigma/A_\sigma$ and $B' = I + \lambda A_\sigma A'$.

Obviously, A' satisfies the conditions of the NTM theorem for $\delta = \sigma - \alpha$. Since $B't \in R_0$ and $1 + \lambda A_\sigma > 0$, it follows that

$$\lim_{n\to\infty} \frac{t_n}{(B't)_n} = \frac{1}{1+\lambda A_\sigma}, \quad \text{i.e.} \quad \lim_{n\to\infty} \frac{s_n}{(Bs)_n} = \frac{1}{1+\lambda A_\sigma}.$$

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