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POSITIVE LOGIC WITH DOUBLE NEGATION

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Abstract. The fragment of the Heyting propositional calculus which contains double negation but does not contain negation is axiomatized by treating double negation as a necessity operator. The resulting system is shown sound and complete with respect to a specific class of Kripke-style models with two accessibility relations, one intuitionistic and the other modal.

0. Introduction. In this paper we shall investigate an intuitionistic propositional calculus $Hdn\Box^+$ for which we shall prove that it is the positive fragment of the system $Hdn\Box$ introduced in K. Došen's paper [**1**]. As $\Box A \leftrightarrow \neg\neg A^-$ holds in $Hdn\Box$, the modal operator $\Box$ can be understood as double negation, so $Hdn\Box^+$ is an answer to the problem, posed in [**1**], of the axiomatization of the fragment of the Heyting propositional calculus H which contains double negation but doesn't contain negation. In other words. since the positive fragment of H is also known as "positive logic", we shall axiomatize positive logic extended with intuitionistic double negation.

1. The syntax of $Hdn\Box^+$. $Hdn\Box^+$ is the propositional calculus in the language $L^+_\Box = \{\rightarrow, \wedge, \vee, \Box\}$ over the set of variables $V = \{p_i | i \in \omega\}$ with the axiom-schemata

$H1$	$A \rightarrow (B \rightarrow A)$
$H2$	$(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$
$H3$	$A \rightarrow (B \rightarrow A \wedge B)$
$H4$	$A \wedge B \rightarrow A$
$H5$	$A \wedge B \rightarrow B$
$H6$	$A \rightarrow A \vee B$
$H7$	$B \rightarrow A \vee B$
$H8$	$(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \vee B \rightarrow C))$
$dn1$	$\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
$dn2$	$A \rightarrow \Box A$
$dn3$	$\Box(A \vee (A \rightarrow B)$
$dn5$	$\Box(\Box A \rightarrow A)$

AMS Subject Classification (1980): Primary 03B45 and the schema-rule

$$MP\frac{A, A \to B}{B}.$$

The system $Hdn\Box$, mentioned in the introduction, is the expansion of $Hdn\Box^+$ in the language $L_\Box = \{\to, \wedge, \vee, \Box, \neg\}$ with the axiom-schemata

$H9$ $(A \to B) \to ((A \to \neg B) \to \neg A)$

$H10$ $\neg A \to (A \to B)$

$dn4$ $\neg\Box\neg(A \to A)$.

Note that our $dn3$ is not the same as $dn3$ in [**1**] where $Hdn\Box$ was first introduced ($\Box(((A \to B) \to A) \to A)$ stands there for $dn3$) but those two formulas are equivalent in $Hdn\Box^+ - dn3$. Also, in $Hdn\Box$, $dn5$ becomes redundant.

2. $Hdn\Box^+$ **frames and models.** The semantics of the intuitionistic modal calculi has been investigated in [**2**]. We shall use the terminology and basic results of that paper.

Definition 1. $\mathcal{H} = \langle X, R_I, R_M\rangle$ is and $Hdn\Box^+$ *frame iff*

(i) $X \neq \emptyset$, $R_I \subseteq X^2$ is reflexive and transitive, $R_M \subseteq X^2$, *and*

(ii) the universal closures of the following first-order formulas hold in $\mathcal{H}$:

(1) xR_Iy *and* $yR_Mz \Rightarrow \exists t(xR_Mt$ *and* $tR_Iz)$

(2) $xR_My \Rightarrow xR_Iy$

(3) xR_My *and* $yR_Iz \Rightarrow zR_Iy$

(5) xR_My *and* $yR_Iz \Rightarrow \exists t(zR_Mt$ *and* $tR_Iz)$

$\operatorname{dom}\mathcal{H} \overset{\text{def}}{=} X$, $KK(Hd\Box^+) \overset{\text{def}}{=} \{\mathcal{H} \mid \mathcal{H}$ is an $Hdn\Box^+$ frame$\}$.

As (1) holds, any Hdn^+ frame is an $Hdn\Box$ frame (see [**2**]). Not that an $Hdn\Box$ frame from [**1**] is an $Hdn\Box^+$ frame which satisfies the universal closure of the first-order formula

$$\exists y \;\; xR_My. \tag{4}$$

In $Hdn\Box$ frames (5) becomes redundant. It is easy to prove, by a simple counterexample, that the class of all $Hdn\Box$ frames $K(Hdn\Box)$ is a proper subclass of $K(Hdn\Box^+)$.

Definition 2. $\mathfrak{M} = \langle \mathcal{H}, v\rangle$ is an $Hdn\Box^+$ *model iff*

(i) $\mathcal{H}$ is an $Hdn\Box^+$ frame, *and*

(ii) *valuation* v maps $\operatorname{dom}\mathcal{H}$ into PV such that

$$xR_Iy \Rightarrow v(x) \subseteq v(y) \tag{her}$$

holds for all $x, y \in \operatorname{dom}\mathcal{H}$.

$$Fr\,\mathfrak{M} \overset{\text{def}}{=} \mathcal{H}, \quad \operatorname{dom}\mathfrak{M} \overset{\text{def}}{=} \operatorname{dom} Fr\,\mathfrak{M},$$

$M(Hdn\Box^+) \stackrel{\text{def}}{=} \{\mathfrak{M} | \mathfrak{M} \text{ is an } Hdn\Box^+ \text{ model}\}$, val $\mathfrak{M} \stackrel{\text{def}}{=} v$.

Validity of formulas in a point of a model, in a model and in a frame are defined as in [**2**], but for the reader not acquainted with that paper we will repeat those definitions restricted, of course, to the formulas in the language $L_\Box^+$. The set of formulas in some language L will be denoted by For(L). Let us note that the following definition is applicable to any class of models satisfying at least the universal closure of the formula (1).

Definition 3. Let $A \in \text{For}(L_\Box^+)$, $\mathfrak{M} \in M(Hdn\Box^+)$ and $x \in \text{dom}\,\mathfrak{M}$.

The predicate Formula A holds in the point x of the $Hdn\Box^+$ *model* $\mathfrak{M}$ ($\langle \mathfrak{M}, x\rangle \models A$, or if there is no ambiguity, only $x \models A$) is defined by recursion on the structure of the formula A. The axioms of this recursion are:

(p_i)	$A = p_i$,	$x \models p_i$	iff	$p_i \in \text{val}\,\mathfrak{M}(x)$
$(\wedge)$	$A = B \wedge C$,	$x \models B \wedge C$	iff	$x \models B$ *and* $x \models C$
$(\vee)$	$A = B \vee C$,	$x \models B \vee C$	iff	$x \models B$ *or* $x \models C$
$(\to)$	$A = B \to C$,	$x \models B \to C$	iff	$\forall y(xR_Iy$ *and* $y \models B \Rightarrow y \models C)$
$(\Box)$	$A = \Box B$,	$x \models \Box B$	iff	$\forall y(xR_My \Rightarrow y \models B)$

A formula A is true in the model $\mathfrak{M}(\mathfrak{M} \models A)$ *iff* $(\forall x \in \text{dom}\,\mathfrak{M})x \models A$

A formula A is valid in the frame $\mathcal{H}(\mathcal{H} \models A)$ *iff* $\forall \mathfrak{M}(Fr\,\mathfrak{M} = \mathcal{H} \Rightarrow \mathfrak{M} \models A)$

A formula A is valid in the class of frames $K(K \models A)$ *iff* $\forall \mathcal{H}(\mathcal{H} \in K \Rightarrow \mathcal{H} \models A)$

For the sake of completeness let us mention that $\models$ for $\neg$ is (in all intuitionistic frames) defined by:

$$(\neg) \quad A = \neg B, \quad x \models \neg B \quad iff \quad \forall y(xR_Iy \Rightarrow \textit{not } y \models B).$$

It is understood that the logic of the meta-connectives *iff* (alternatively: $\Leftrightarrow$), *and or, not*, $\Rightarrow$, $\forall$ and $\exists$ is classical and that the domain of the quantifiers $\forall$ and $\exists$ when they stand in front of the individual variables $x, y, z, t, \dots$ is $dom\mathfrak{M}$.

The property (her) $xR_Iy \Rightarrow v(x) \subseteq v(y)$ of the valuation can be transformed (because of the definition above) into $xR_Iy \Rightarrow (x \models p_i \Rightarrow y \models p_i)$. This properly extends to all formulas (for a proof see (2); only property (1) of $Hdn\Box^+$ frames matters) namely the following holds:

Intuitionistic Heredity (Her). *For any* $A \in For(L_\Box^+), \mathfrak{M} \in M\ (Hdn\Box^+)$ *and* $x, y \in \text{dom}\,\mathfrak{M}$

$$\text{(Her)} \qquad xR_Iy \Rightarrow (x \models A \Rightarrow y \models A).$$

3. Completeness of $Hdn\Box^+$. In this section we are going to prove our main theorem, which states that

Theorem 1. $Hdn\Box^+ \vdash A \Leftrightarrow K(Hdn\Box^+) \models A$.

First, we are going to prove the following

SOUNDNESS THEOREM for $Hdn\Box^+$. $Hdn\Box^+ \vdash A \Rightarrow K(Hdn\Box^+) \models A$.

In proving this we will use the following lemma:

LEMMA 1. *For any $H\Box$ frame $\mathcal{H}$,*

1.1 $\mathcal{H} \models \Box(A \vee (A \rightarrow B)) \Leftrightarrow \mathcal{H} \models \forall x \forall y \forall z(xR_My \text{ and } yR_Iz \Rightarrow zR_Iy)$

1.2 $\mathcal{H} \models \Box(\Box A \rightarrow A) \Leftrightarrow \mathcal{H} \models \forall x \forall y \forall z(xR_My \text{ and } yR_z \Rightarrow$

$\Rightarrow \exists t(zR_Mt \text{ and } tR_Iz))$.

Proof. Because of Definition 3, when proving $\mathcal{H} \models A$ for some propositional formula A, it suffices to prove $x \models A$ for an arbitrary $x \in \operatorname{dom} \mathcal{H}$ for an arbitrary model (valuation) on $\mathcal{H}$. In the following proofs we are omitting universal quantifiers which range over whole formulas.

1.1 ($\Leftarrow$) Suppose $\mathcal{H} \models \forall x \forall y \forall z$ (3) and let $\mathfrak{M}$ be model on $\mathcal{H}$ and $x \in \operatorname{dom} \mathcal{H}$.

$x \models \Box(A \vee (A \rightarrow B))$

iff $xR_My \Rightarrow (y \models A \text{ or } y \models A \rightarrow B)$

iff $xR_My \text{ and } (\text{not } y \models A) \Rightarrow (yR_Iz \text{ and } z \models A \Rightarrow z \models B)$

iff $xR_My \text{ and } (\text{not } y \models A) \text{ and } yR_Iz \text{ and } z \models A \Rightarrow z \models B$.

The last formula is true since its antecedent is false: xR_My *and* yR_Iz implies (by (3)) zR_Iy which, as $z \vdash A$, implies (by Her) $y \vdash A$ which contradicts not $y \models A$.

($\Rightarrow$) Suppose not $\mathcal{H} \models \forall x \forall y \forall z$ (3); then there exist $a, b, c \in \operatorname{dom} \mathcal{H}$ such that

$$aR_Mb \text{ and } bR_Ic \text{ and } (\text{not } cR_Ib).$$

Define $v : \operatorname{dom} \mathcal{H} \rightarrow PV$ with $p_0 \in v(t) \Leftrightarrow (\text{not } tR_Ib, p_1 \notin v(t)$ and $p_i \in v(t)$ for all $i \neq 0, 1$. It is easy to prove that v is a valuation and that in the model $\langle \mathcal{H}, v \rangle$ *not* $a \models \Box(p_0 \vee (p_0 \rightarrow P_1))$ as aR_Mb and *not* $b \models p_0$ and *not* $b \models p_0 \rightarrow p_1$ (as bR_Ic and $c \models p_0$ and *not* $c \models p_1$).

1.2 ($\Leftarrow$) Suppose $\mathcal{H} \models \forall x \forall y \forall z$ (5) and let $\mathfrak{M}$ be a model on $\mathcal{H}$ and $x \in \operatorname{dom} \mathfrak{M}$.

$x \models \Box(\Box A \rightarrow A)$

iff $xR_My \Rightarrow y \models \Box A \rightarrow A$

iff $xR_My \text{ and } yR_Iz \text{ and } z \models \Box A \Rightarrow z \models A$.

The last formula is true since xR_My *and* yR_Iz implies (by (5)) zR_Mt *and* tR_Iz for some t, for which, as $z \models \Box A$, we have $t \models A$, and, as tR_Iz, by Her, $z \models A$ also.

($\Rightarrow$) Suppose *not* $\mathcal{H} \models \forall x \forall y \forall z$ (5), then there exist $a, b, c \in \operatorname{dom} \mathcal{H}$ such that

$$aR_Mb \text{ and } bR_Ic \text{ and } \forall t(cR_Mt \Rightarrow \text{not } tR_Ic).$$

Define $v : \operatorname{dom} \mathcal{H} \rightarrow PV$ with $p_0 \in v(t) \Leftrightarrow \text{not } tR_Ic$ and $p_i \in v(t)$ for all $i \neq 0$. It is easy to prove that v is a valuation and that in the model $\langle \mathcal{H}, v \rangle$, *not* $a \models \Box(\Box p_0 \rightarrow p_0)$ as aR_Mb, and *not* $b \models p_0 \rightarrow p_0$ as bR_Ic and $c \models \Box p_0$ (as cR_Mt implies *not* tR_Ic i.e. $t \models p_0$), and *not* $c \models p_0$.

The proof of the theorem for $Hdn\Box^+$ is complete now as the soundness for the schemata H1 – H8 and the rule MP follows from any standard proof of the soundness of the Heyting calculus H with respect to the Kripke frames for H: our $Hdn\Box^+$ frames are a special kind of those (see, for example, [**3**]); the soundness for the scemata $dn1$ and $dn2$ follows from [**1**] where it has been proved that $dn1$ ($dn2$) holds in a $H\Box$ frame iff in this frame the universal closure of (1) ((2)) holds which is the case; the soundness of the schemata (3) and (5) follows from Lemma 1.

To prove Theorem 1 we have to prove $K(Hdn\Box^+) \models A \Rightarrow Hdn\Box^+ \vdash A$. For that we use the technique of canonical frames and models which has been introduced in [**2**]. First, we are defining the basic notions.

Definition 4. Let $x, y \subseteq \mathrm{For}\,(L_\Box^+)$ and $A, B \in L(L_\Box^+)$.
$x \vdash_{Hdn\Box^+} A$ *iff* there is a proof for A in $Hdn\Box^+ \cup x$.
x is $Hdn\Box^+$ *deductively closed iff* $\forall A(x \vdash_{Hdn\Box^+} A \Rightarrow A \in x)$.
x is *prime iff* $\forall A \forall B(A \vee B \in x \Rightarrow A \in x$ *or* $B \in x)$.
x is *consistent iff not* $\forall A x \vdash_{Hdn\Box^+} A$.
x is $Hdn\Box^+$ *nice iff* x is prime, consistent and $Hdn\Box^+$ deductively closed.
$\mathcal{H}^c(Hdn\Box^+) \stackrel{\text{def}}{=\!=} \langle X^c(Hdn\Box^+), R_I^c, R_M^c\rangle$ is *canonical* $Hdn\Box^+$ *frame iff*

(i) $X^c(Hdn\Box^+)$ is the set of all $Hdn\Box^+$ nice sets of formulas
(ii) $xR_I^c y \Leftrightarrow x \subseteq y$
(iii) $xR_M^c y \Leftrightarrow x_\Box \subseteq y$, where $x_\Box \stackrel{\text{def}}{=\!=} \{A|\,\Box A \in x\}$.

$\mathfrak{M}(Hdn\Box^+) \stackrel{\text{def}}{=\!=} \langle \mathcal{H}^c(Hdn\Box^+), vc\rangle$ is the *canonical* $Hdn\Box^+$ *model iff*

$$v^c(x) = x \cap V \text{ for all } x \subseteq X^c(Hdn\Box^+).$$

LEMMA 2. $\mathcal{H}^c(Hdn\Box^+)$ and $\mathfrak{M}^c(Hdn\Box^+)$ *are an* $Hdn\Box^+$ *frame and model.*

Proof. v^c is obviously a valuation as R_I^c coincides with set-theoretic inclusion. It remains to prove that $\mathcal{H}^c(Hdn\Box^+)$ is an $Hdn\Box^+$ frame. The relation $R_I^c(\subseteq)$ is reflexive and transitive, (1) is a trivial set-theoretic consequence of the definition of the relations R_I^c and R_M^c, and (2) is proved (in [**1**]) to be a consequence of $dn2$. It remains to prove that $X^c(Hdn\Box^+) \neq \emptyset$ and that in the canonical frame (3) and (5) hold. $X^c(Hdn\Box^+) \neq \emptyset$ because the set of all theorems of $Hdn\Box^+$ is nice, since it is deductively closed by definition consistent, since it is a subset of the set of all theorems of the Heyting calculus H (if $\Box$ is interpreted as $\neg\neg$), and it is prime, as can be shown by standard methods.

(3) $\quad xR_M y$ *and* $yR_I z \Rightarrow zR_I y$ in a canonical frame becomes
$\quad x_\Box \subseteq y$ *and* $y \subseteq z \Rightarrow z \subseteq y$.

Suppose $x_\Box \subseteq y$ *and* $y \subseteq z$ and let $A \in z$. We prove that $A \in y$. As z is consistent, there is a B such that $B \notin z$. As $\Box(A \vee (A \to B))$ is a theorem of $Hdn\Box^+$, it belongs to x and, consequently, $A \vee (A \to B) \in x_\Box \subseteq y$. As y is prime $A \in y$ *or* $A \to B \in y$. If $A \to B \in y$, because if $y \subseteq z$, $A \to B \in z$ which, with $A \in z$, gives $B \in z$; but $B \notin z$. So $A \in y$.

$$xR_My \ \textit{and} \ yR_Iz \Rightarrow \exists t(zR_Mt \ \textit{and} \ tR_Iz) \tag{5}$$

in the canonical frame becomes

$$x_\Box \subseteq y \ \textit{andy} \subseteq z \Rightarrow \exists t(z_\Box \subseteq t \ \textit{and} \ t \subseteq z).$$

It suffices to prove $x_\Box \subseteq z \Rightarrow z_\Box \subseteq z$.

Suppose $x_\Box \subseteq z$ and let $A \in z_\Box$, i.e. $\Box A \in z$. Because of $dn5$ $\Box(\Box A \to A) \in x$ and $\Box A \to A \in x_\Box \subseteq z$ which, together with $\Box A \in z$ implies $A \in z$.

LEMMA 3. *For any $A \in For(L^+_\Box)$ and any $Hdn\Box^+$ nice set of formulas x*

$$\langle \mathfrak{M}^c(Hdn\Box^+), x\rangle \models A \Leftrightarrow A \in x.$$

Proof. As in the appropriate proof for canonical models of the calculus $HK\Box$ in [**2**]. Note that $HK\Box$ contains negation but as the proof goes by induction on the number of connectives in the formula A, the step $(\neg)$ should be simply omitted. The steps in the proof for the positive connectives ($\to$, $\vee$, $\vee$,$\Box$) rely only on the schemata H1 – H8, $dn1$ and rule MP – and this is a part of the calculus $Hdn\Box^+$.

Because of previous lemmata:

$$\textit{not} \ \ Hdn\Box^+ \vdash A \Rightarrow \exists x(x \ \text{is} \ Hdn\Box^+ \ \text{nice} \ \textit{and} \ A \in x).$$

The implication above is true if we take the set of theorems of $Hdn\Box^+$ for x. Next we have

$$\begin{aligned} \textit{not} \ \ Hdn\Box^+ \vdash A &\Rightarrow \exists x \langle \mathfrak{M}^c(Hdn\Box^+), x\rangle \not\models A \quad &&\text{(by Lemma 3)} \\ &\Rightarrow K(Hdn\Box^+) \not\models A &&\text{(by Lemma 2).} \end{aligned}$$

4. $Hdn\Box^+$ is the positive fragment of $Hdn\Box$. In order to prove this we introduce a special kind of expansions of $Hdn\Box+$ frames and models.

Definition 5. Let $\mathcal{H} = \langle X, R_I, R_M\rangle$ be a structure with two binary relations R_I and R_M on a nonempty domain X and let $1 \notin X$. $\overline{H} \stackrel{\text{def}}{=\!=} \langle \overline{X}, \overline{R}_I, \overline{R}_M\rangle$ is a *closure of $\mathcal{H}$ iff*

(i) $\overline{X} = X \cup \{1\}$

(ii) $x\overline{R}_Iy \Leftrightarrow xR_Iy$ *or* $(\exists z(xR_Iz$ *and not* $\exists tzR_Mt)$ *and* $y = 1)$ *or* $x = y = 1$

(iii) $xR_My \Leftrightarrow xR_MIy$ *or* $(x\overline{R}_I1$ *and* $y = 1)$.

LEMMA 4. *The closure of an $Hdn\Box^+$ frame is an $Hdn\Box$ frame.*

Proof. We are going to prove that in $\mathcal{H}$, where $\mathcal{H}$ is an $Hdn\Box+$ frame, the relation $\overline{R}_I$ is reflexive and transitive and (1) – (4) hold. (5) is redundant as it follows from (4). Furthermore, as $\overline{R}_I/X^2 = R_I$ and $\overline{R}_M/X^2 = R_M$, reflexivity and transitivity of $\overline{R}_I$, as well as (1), (2) and (3) are to be proved only in case when at least one of the free variables takes the value 1; in other cases they are already true in $\mathcal{H}$.

First, we list some trivial but useful properties of $\overline{R}_I$, and $\overline{R}_M$.

(I1) $\quad 1\overline{R}_I x \Leftrightarrow x = 1$
(I2) $\quad x\overline{R}_I 1 \Leftrightarrow x = 1 \ \textit{or} \ \exists z(xR_Iz \ \textit{and} \ (\textit{not} \ \exists tzR_Mt))$
(M1) $\quad 1\overline{R}_M x \Leftrightarrow x = 1$
(M2) $\quad x\overline{R}_M 1 \Leftrightarrow x\overline{R}_I 1.$

$\overline{R}_I$ is reflexive: $x\overline{R}_I x \Leftrightarrow x\overline{R}_I x \ \textit{or} \ x = 1$, which is true.

R_I is transitive $x\overline{R}_I y \ \textit{and} \ y\overline{R}_I z \Rightarrow x\overline{R}_I z$.

Suppose $x\overline{R}_I y \ \textit{and} \ y\overline{R}_I z$. If $x = 1$, then by (I1) $y = z = 1$, so $x\overline{R}_I z$ is true. If $y = 1$ then, by (I1) again, $z = 1$, so $x\overline{R}_I z$ reduces to $x\overline{R}_I y$. Suppose $x \neq 1$, $y \neq 1$ and $z = 1$; then

$$\begin{aligned}
x\overline{R}_I y \ \textit{and} \ y\overline{R}_I z &\Rightarrow xR_I y \ \textit{and} \ y\overline{R}_I 1 \\
&\Rightarrow xR_I y \ \textit{and} \ \exists z(yR_Iz \ \textit{and} \ (\textit{not} \ \exists tzR_Mt)) && \text{(By (I2))} \\
&\Rightarrow \exists z(xR_Iz \ \textit{and} \ (\textit{not} \ \exists tzR_Mt)) && \text{(as } R_I \text{ is transitive)} \\
&\Rightarrow x\overline{R}1 && \text{(by (I2) again)} \\
&\Rightarrow x\overline{R}_I z && \text{(as } z = 1).
\end{aligned}$$

$$x\overline{R}_I y \ \textit{and} \ y\overline{R}_M z \Rightarrow \exists t(x\overline{R}_M t \ \textit{and} \ t\overline{R}_I z). \tag{1}$$

Suppose $x\overline{R}_I y \ \textit{and} \ y\overline{R}_M z$. If $x = 1$, then, by (I1), $y = 1$ and, by (M1), $z = 1$. It suffices to take $t = 1$. If $y = 1$, then, by (M1), $z = 1$ and $x\overline{R}_I y$ reduces to $x\overline{R}_I 1$. But by (M2) the last is equivalent with $x\overline{R}_M 1$. As $z = 1$, we have $1\overline{R}_I z$ (by (I1)). So, it suffices to take $t = 1$. If $z = 1$, then $y\overline{R}_M z$ reduces to $y\overline{R}_M 1$ and, by (M2), to $y\overline{R}_I 1$. But, as we have proved, $\overline{R}_I$ is transitive; so $x\overline{R}_I y \ \textit{and} \ y\overline{R}_I 1$ implies $x\overline{R}_I 1$ and, again by (M2), $x\overline{R}_M 1$. As $z = 1$, it suffices to take $t = 1$.

$$x\overline{R}_M y \Rightarrow x\overline{R}_I y. \tag{2}$$

Suppose $x\overline{R}_M y$. If $x = 1$, $x\overline{R}_M y$ reduces to $1\,\overline{R}_M y$; so, by (M1), $y = 1$ and $x\overline{R}_I y$ holds. If $y = 1$, $x\overline{R}_M y$ reduces to $x\overline{R}_M 1$ and, by (M2), to $x\overline{R}_I 1$. So $x\overline{R}_I y$ holds again.

$$x\overline{R}_M y \ \textit{and} \ y\overline{R}_I z \Rightarrow z\overline{R}_I y. \tag{3}$$

Suppose $x\overline{R}_M y \ \textit{and} \ y\overline{R}_I z$. If $x = 1$, then $y = 1$ by (M1) and $z = 1$ by (I1). So, $z\overline{R}_I y$. If $y = 1$, then $z = 1$ by (I1), so $z\overline{R}_I y$ again. If $x \neq 1$, $y \neq 1$ and $z = 1$ the antecedent of (3) reduces to

$$xR_M y \ \textit{and} \ \exists z(yR_Iz \ \textit{and} \ (\textit{not} \ \exists tzR_Mt)) \qquad \text{(by (I2))}.$$

But the last formula is a contradiction as by (5), x, y, z being in the former $Hdn\Box^+$ frame, xR_My *and* yR_Iz implies $\exists tzR_Mt$. As the antecedent of formula (3) is false in the case $z = 1$, $x \neq 1$, $y \neq 1$, this formula is true in that case.

Note that this is the only place in the proof where condition (5) is used.

(4) $$\exists yx\overline{R}_My.$$

If $x = 1$, (5) reduces to $\exists y1\overline{R}_My$ which is true for $y = 1$ by (M1). Formula (5) doesn't necessarily hold in $Hdn\Box^+$ frames, so it should be checked for $x \neq 1$. Let $x \neq 1$. If $\exists yxR_My$ then, because $R_M \subseteq \overline{R}_M$, $\exists yx\overline{R}_My$, also holds. If *not* $\exists txR_Mt$, then, as R_I is reflexive, xR_Ix *and* (*not* $\exists txR_Mt$) holds. But that means that $\exists z(xR_Iz$ *and* (*not* $\exists tzR_Mt))$ also holds (take $z = x$). The last formula implies $x\overline{R}_M1$ by (I2) and (M2).

This ends the proof.

Definition 6. Let $\mathfrak{M}$ be an $Hdn\Box^+$ model. $\overline{\mathfrak{M}} \overset{\text{def}}{=\!=} \langle \overline{Fr\mathfrak{M}}, \overline{v}\rangle$ is a closure of the model $\mathfrak{M}$ iff $\overline{v}/\mathrm{dom}\,\overline{Fr\mathfrak{M}} = \mathrm{val}\,\mathfrak{M}$ and $\overline{v}(1) = V$.

As $\mathrm{val}\,\mathfrak{M}(x) \subseteq \overline{v}(1)$ for all $x \in \mathrm{dom}\,\overline{Fr\mathfrak{M}}$, the following holds:

LEMMA 5. *The closure of an $Hdn\Box^+$ model is an $Hdn\Box$ model.*

The following lemma shows that, with respect to positive formulas, $Hdn\Box^+$ models and their closures have the same properties.

LEMMA 6. *Let $\mathfrak{M}$ be an $Hdn\Box^+$ model and $\overline{\mathfrak{M}}$ its closure. Then the following holds for all $A \in For(L^+_\Box)$.*

6.1 $\langle \overline{\mathfrak{M}}, 1\rangle \models A$

6.2 $\langle \overline{\mathfrak{M}}, x\rangle \models A$ *iff* $\langle \mathfrak{M}, x\rangle \models A$, *for all* $x \in \mathrm{dom}\,\mathfrak{M}$.

Proof. We are going to prove 6.1 and 6.2 by induction on the number $s(A)$ of connectives in the formula A. If $s(A) = 0$, then $A = p_i$ for some $i \in \omega$, so

$(p_i) A = p_i$,

$\langle \overline{\mathfrak{M}}, 1\rangle \models p_i$	*iff*	$p_i \in \overline{v}(1) = V$,	which is true for all $i \in \omega$
$\langle \overline{\mathfrak{M}}, x\rangle \models p_i$	*iff*	$p_i \in \overline{v}(x) = \mathrm{val}\,\mathfrak{M}(x)$	(As $x \in \mathrm{dom}\,\mathfrak{M}$)
	iff	$\langle \overline{\mathfrak{M}}, x\rangle \models p_i$.	

Let $s(A) > 0$ and let 6.1 *and* 6.2 $\overset{\text{def}}{\Longleftrightarrow}$ Ind Hyp hold for all formulas F such that $s(F) < s(A)$. The following may occur:

$(\wedge)$ $a = b \wedge C$,

$\langle \overline{\mathfrak{M}}, 1\rangle \models B \wedge C$	*iff*	$\langle \overline{\mathfrak{M}}, 1\rangle \models B$	*and*	$\langle \overline{\mathfrak{M}}, 1\rangle \models C$	– true by Ind Hyp
$\langle \overline{\mathfrak{M}}, x\rangle \models B \wedge C$	*iff*	$\langle \overline{\mathfrak{M}}, x\rangle \models B$	*and*	$\langle \overline{\mathfrak{M}}, x\rangle \models C$	
	iff	$\langle \mathfrak{M}, x\rangle \models B$	*and*	$\langle \mathfrak{M}, x\rangle \models C$	(by Ind Hyp)
	iff	$\langle \mathfrak{M}, x\rangle \models B \vee C$.			

$(\vee)$ $A = B \vee C$,

$\langle \overline{\mathfrak{M}}, 1\rangle \models B \vee C$	*iff*	$\langle \overline{\mathfrak{M}}, 1\rangle \models C$	*or*	$\langle \overline{\mathfrak{M}}, 1\rangle \models C$	– true by Ind Hyp

$$\begin{array}{llll} \langle\overline{\mathfrak{M}},x\rangle \models B\vee C & \textit{iff} & \langle\overline{\mathfrak{M}},x\rangle \models B \quad \textit{or} \quad \langle\overline{\mathfrak{M}},x\rangle \models C & \\ & \textit{iff} & \langle\mathfrak{M},x\rangle \models B \quad \textit{or} \quad \langle\mathfrak{M},x\rangle \models C & \text{(by Ind Hyp)} \\ & \textit{iff} & \langle\mathfrak{M},x\rangle \models B\vee C. & \end{array}$$

$(\to)$ $A = B \to C$

$$\begin{array}{lll} \langle\overline{\mathfrak{M}},1\rangle \models B\to C & \textit{iff} & \forall y(1\overline{R}_I y \ \textit{and}\ \langle\overline{\mathfrak{M}},y\rangle \models B \Rightarrow \langle\overline{\mathfrak{M}},y\rangle \models C) \\ & \textit{iff} & \langle\overline{\mathfrak{M}},1\rangle \models B \Rightarrow \langle\overline{\mathfrak{M}},1\rangle \models C \qquad (\text{as } 1\overline{R}_I y \Leftrightarrow y=1). \end{array}$$

The last formula is true as its consequent is true by Ind Hyp.

$$\begin{array}{lll} \langle\overline{\mathfrak{M}},x\rangle \models B\to C & \textit{iff} & \forall y(x\overline{R}_I y \ \textit{and}\ \langle\overline{\mathfrak{M}},y\rangle \models B \Rightarrow \langle\overline{\mathfrak{M}},y\rangle \models C) \\ & \textit{iff} & \forall y(x\overline{R}_I y \ \textit{and}\ y\neq 1 \ \textit{and}\ \langle\overline{\mathfrak{M}},y\rangle \models B \Rightarrow \langle\overline{\mathfrak{M}},y\rangle \models C) \\ & \textit{and} & \forall y(x\overline{R}_I y \ \textit{and}\ y= 1 \ \textit{and}\ \langle\overline{\mathfrak{M}},y\rangle \models B \Rightarrow \langle\overline{\mathfrak{M}},y\rangle \models C) \\ & \textit{iff} & \forall y(x\overline{R}_I y \ \textit{and}\ \langle\mathfrak{M},y\rangle \models B \Rightarrow \langle\mathfrak{M},y\rangle \models C) \\ & \textit{and} & \forall y(x\overline{R}_I 1 \ \textit{and}\ \langle\overline{\mathfrak{M}},1\rangle \models B \Rightarrow \langle\overline{\mathfrak{M}},1\rangle \models C). \end{array}$$

The last step is true by Ind Hyp as $y \neq 1$ in the first conjunct, and the second conjunct is true as its consequent is true by Ind Hyp. Hence,

$\langle\overline{\mathfrak{M}},x\rangle \models B\to C \quad \textit{iff} \quad \langle\mathfrak{M},x\rangle \models B\to C.$

$(\Box)$ $A = \Box B$,

$$\begin{array}{llll} \langle\overline{\mathfrak{M}},1\rangle \models \Box B & \textit{iff} & \forall y(1\overline{R}_M y \Rightarrow \langle\overline{\mathfrak{M}},y\rangle \models B) & \\ & \textit{iff} & \langle\overline{\mathfrak{M}},1\rangle \models B & (\text{as } 1\overline{R}_M y \Leftrightarrow y=1) \end{array}$$

The last formula is true by Ind Hup.

$$\begin{array}{lll} \langle\overline{\mathfrak{M}},x\rangle \models \Box B & \textit{iff} & \forall y(x\overline{R}_M y \Rightarrow \langle\overline{\mathfrak{M}},y\rangle \models B) \\ & \textit{iff} & \forall y(x\overline{R}_M y \ \textit{and}\ y\neq 1 \Rightarrow \langle\overline{\mathfrak{M}},y\rangle \models B) \\ & \textit{and} & \forall y(x\overline{R}_M y \ \textit{and}\ y= 1 \Rightarrow \langle\overline{\mathfrak{M}},y\rangle \models B) \\ & \textit{iff} & \forall y(xR_M y \Rightarrow \langle\mathfrak{M},y\rangle \models B) \\ & \textit{and} & \forall y(1\overline{R}_M 1 \Rightarrow \langle\overline{\mathfrak{M}},1\rangle \models B). \end{array}$$

The last step is true by Ind Hyp as $y \neq 1$ in the first conjunct, and the second conjunct is true as its consequent is true by Ind Hyp. Hence,

$\langle\overline{\mathfrak{M}},x\rangle \models \Box B \quad \textit{iff} \quad \langle\mathfrak{M},x\rangle \models \Box B.$

This ends the proof.

Finally we have:

THEOREM 2. *$K(Hdn\Box^+) \models A \Leftrightarrow K(Hdn\Box) \models A$, for all $A \in \mathrm{For}\,(L^+_\Box)$.*

Proof. The $(\Rightarrow)$ part of the proof follows trivially from $K(Hdn\Box) \subseteq K(Hdn\Box^+)$.

$(\Leftarrow)$ We prove the contraposition of the statement. Suppose that A is not valid in the class $K(Hdn\Box^+)$. Then there exists an $Hdn\Box^+$ model $\mathfrak{M}$ and $x \in \mathrm{dom}\,\mathfrak{M}$ such that $\langle\mathfrak{M},x\rangle \not\models A$. By the previous lemma, as $x \in \mathrm{dom}\,\mathfrak{M}$, $\langle\overline{\mathfrak{M}},x\rangle \not\models A$. By Lemma 4, $\overline{\mathfrak{M}}$ is an $Hdn\Box$ model, so A is not valid in the class $K(Hdn\Box)$.

From Theorem 2 and from [**1**], where it has been proved that $K(Hdn\Box) \models A \Leftrightarrow Hdn\Box \vdash A$, we, obtain $Hdn\Box \vdash A \Leftrightarrow K(Hdn\Box+) \models A$ for all $A \in \mathrm{For}\,(L^+_\Box)$.

But from our Theorem 1 we have that $K(Hdn\Box+) \models A \Leftrightarrow Hdn\Box^+ \vdash A$, so we have:

THEOREM 3. *$Hdn\Box^+ \vdash A \Leftrightarrow Hdn\Box \vdash A$, for all $A \in \mathrm{For}\,(L_s quare^+)$ i.e. $Hdn\Box^+$ is the positive fragment of $Hdn\Box$, and, if $\Box$ is interpreted as $\neg\neg$, $Hdn\Box^+$ is the positive fragment of the Heyting calculus H with double negation.*

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