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ON TWO OPEN PROBLEMS OF CONTRACTIVE MAPPINGS

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Abstract. Two open problems are solved concerning the fixed points of contractive mappings. The first is an example of a shrinking mapping of the closed unit ball in a Banach space without any fixed point. The second solves a question of B. Fischer.

1. Let (X, d) be a metric space, $T : X \to X$ a mapping of X into itself. T is said to be shrinking if $d(Tx, Ty) < d(x, y)$ for every $x, y \in X$.

It is well known (see e.g. [**3**]) that if X is compact and $T : X \to X$ is a shrinking mapping, then T has a fixed point. By a beautiful theorem of Browder [**1**] the same conclusion holds provided X is the closed unit ball of a Hilbert space and T is shrinking. In connection with these results D. R. Smart raised the following question [**3**, p. 39]: "Does every shrinking mapping of the closed unit ball in a Banach space have a fixed point?" The aim of this paragraph is to give negative answer to this problem.

THEOREM 1. *There exists a Banach space B and an affine shrinking mapping T of the closed unit ball U of B into the boundary ∂U of U such that T does not have any fixed point.*

Proof. Let $c_0 = \{x = \{x_i\}_{i=1}^{\infty} \mid \lim_{i\to\infty} x_i = 0\}$ be the space real sequences converging to 0 with norm $\|x\| = \sup_i \mid x_i \mid$. Let $B = c_0$ and $T(x_1, x_2, \dots, x_n, \dots) = (1, x_2/2+1/2,\ 2x_2/3+1/3, \dots,\ (1-1/n)x_n+1/n, \dots)$ i.e. T is defined by $(Tx)_n) = (1-1/n)x_n+1/n$. If U is the unit ball in B, then clearly $T : U \to \partial U$ and T is affine. T is shringing. Let $x = \{x_i\}_1^{\infty}$, $y = \{y_i\}_1^{\infty}$, $x \neq y$. Then

$$0 < \varepsilon := \|x - y\| = |x_{n_0} - y_{n_0}|$$

for some n_0. Let $N > 2$ be so large that the inequalities

$$|x_n| < \varepsilon/4, \ |y_n| < \varepsilon/4 \qquad (n \geq N)$$

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be satisfied. Now

$$|(Tx)_i - (Ty)_i| = (1-1/i)|x_i - y_i| \le \begin{cases} 2\,\varepsilon/4 = \varepsilon/2 & \text{if } i \ge N \\ (1-1/N)|x_i - y_i| \le |1-1/N|\varepsilon & \text{if } i < N \end{cases}$$

i.e.

$$\|Tx - Ty\| \le (1-1/N)\varepsilon,$$

and so T is really a shrinking mapping.

Finally T does not have any fixed point: if $x = \{x_i\}_1^\infty$ where a fixed point of T, then we would have

$$(1-1/i)x_i + 1/i = (Tx)_i = x_i$$

i.e. $x_i = 1$ for all i, but the sequence $\{1\}_1^\infty$ does not belong to $B = c_0$.

We have proved our theorem.

2. In [**2**] B. Fischer made the following conjecture. Suppose S and T are mapping of the complete matrix space X into itself, with either S or T continuous, satisfying the inequality

$$d(Sx, TSy) \le c\,\mathrm{diam}\{x, Sx, Sy, TSy\} \tag{1}$$

for all x, y in X, where $0 \le c < 1$. Then S and T have a unique common fixed point.

This conjecture has been open even for compact X. Now we show that it is true for $c < 1/2$ but false for $c \ge 1/2$.

THEOREM 2. *If X is complete, $S : X \to X$, $T : X \to X$ with property* (1), *where $c < 1/2$, then S and T have a unique common fixed point. On the other hand, there are a four point X and $S : X \to X$, $T : X \to X$ mappings of X without fixed point satisfying*

$$d(Sx,\ TSy) \ge 1/2\ \mathrm{diam}\ \{x,\ Sx,\ Sy,\ TSy\}.$$

Thus, if $\alpha < 1/2$ we do not need any continuity assumption, and for $\alpha \ge 1/2$ even the simultaneous continuity of S and T and the compactness of X do not help.

Proof. To prove the first part of our theorem let $x_0 \in X$ be arbitrary and let

$$x_n = \begin{cases} (TS)^{n/2}x_0, & \text{if } n \text{ is even} \\ S(TS)^{(n-1)/2}x_0, & \text{if } n \text{ is odd.} \end{cases}$$

By (1)

$$d(x_{2n+1}, x_{2n}) = d(STSx_{2n-2}, TSx_{2n-2}) \le c\,\mathrm{diam}\{Sx_{2n-2}, TSx_{2n-2}, STSx_{2n-2}\} =$$
$$= c\,\mathrm{diam}\{x_{2n-1},\ x_{2n},\ x_{2n+1}\} \le c(d(x_{2n}, x_{2n-1}) + d(x_{2n+1}, x_{2n}))$$

and thus

$$d(x_{2n+1}, x_{2n}) \le (c/(1-c))d(x_{2n}, x_{2n-1}) \qquad (n \ge 1) \tag{2}$$

Similarly,

$$d(x_{2n+2}, x_{2n+1}) = d(Sx_{2n}, TSx_{2n}) \le c \ \operatorname{diam}\{x_{2n}, x_{2n+1}, x_{2n+2}\} \le$$
$$\le c(d(x_{2n+1}, x_{2n}) + d(x_{2n+2}, x_{2n+1}))$$

by which

$$d(x_{2n+2}, x_{2n+1}) \le (c/(1-c))d(x_{2n+1}, x_{2n}) \tag{3}$$

Since $c < 1/2$ we have $c/(1-c) < 1$, and so (2) and (3) imply that the sequence x_n is a Cauchy sequnce and thus, by completeness, $x_n \to z(n \to \infty z \in X)$. Using again (1) we get

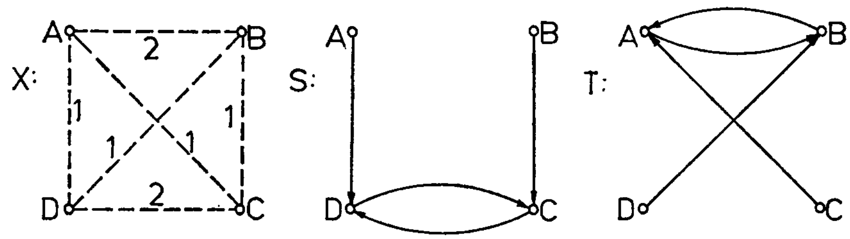
$$d(Sz, x_{2n+2}) \le c \ \operatorname{diam}\{z, Sz, x_{2n+1}, x_{2n+2}\} \le$$
$$c(d(Sz, z) + d(z, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}))$$

Letting here $n \to \infty$ we obtain $d(Sz, z) \le cd(Sz, z)$ i.e. $d(Sz, z) = 0$, $Sz = z$. But then

$$d(z, Tz) = d(Sz, TSz) \le c \ \operatorname{diam}\{z, Sz, TSz\} = c\, d(z, Tz)$$

i.e. $d(z, Tz) = 0$, $Tz = z$ and thus z is a common fixed point of S and T. The uniqueness of the common fixed point follows easily from (1).

After this let us prove that the conjecture is false for $c = 1/2$ and hence also $c \ge 1.2$. Let $X = \{A, B, C, D\}$ with $d(A, D) = d(B, C) = d(B, D) = 1$ and $d(A, B) = d(C, D) = 2$ (see the first figure) and let S and T be the two mapping indicated below:



Neither S nor T have any fixed point. However, $Sx \in \{D, C\}$, $TSy \in \{A, B\}$ and so $d(Sx, TSy) = 1$ for every $x, y \in X$; furthermore

a) $d(x, Sx) = 2$, if $x = C$ or $x = D$

b) $d(Sx, Sy) = 2$, if $x + A$ and $y \in \{B, D\}$ or $x = B$ and $y \in \{A, C\}$

c) $d(x, TSy) = 2$, if $x = A$ and $y \in \{A, C\}$or $x = B$ and $y \in \{B, D\}$, i.e. in any case $\operatorname{diam}\{x, Sx, Sy, TSy\} = 2$ and so (1) holds for every $x, y \in X$ with $c = 1/2$.

We have proved our theorem.

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