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ON PARA-A-EISTEIN MANIFOLDS

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1. Introduction. An n-dimensional smooth manifold M_n with a tensor field f of type (1.1.), a vector field T, a l-form A and a Riemannian matric g is said to be an almost paracontact Riemannian manifold if [**2**]

(1.1) $\quad (a)\ f^2 = I - A \otimes T$

(1.1) $\quad (b)\ g(fX, fY) = g(X,Y) - A(X)A(Y).$

It can be shown that

(1.2) $\quad A(T) = 1,\ \ fT = 0,\ \ Af = 0,\ \ \operatorname{rank}(f) = n - 1$

whence it follows that

$$g(T,T) = 1,\ \ 'f(X,Y) =' f(Y,X)$$

$'f$ being defined by $'f(X,Y) = g(fX,Y)$ and X,Y standing for arbitrary vector fields on M_n.

If D be the Riemannian connexion induced on M_n by g such that [**3**]

(1.3) $$(D_X A)Y + (D_Y A)X = 2'f(X,Y)$$

then the almost paracontact Riemannian manifold M_n is termed a paracontact Riemannian manifold. A paracontact Riemannian manifold M_n whose l-form A is closed, that is

(1.4) $$(D_X A)Y - (D_Y A)X = 0$$

(1.5) $$(D_X f)Y = 2A(X)A(Y)T - g(X,Y)T - A(Y)X$$

is called a normal paracontact Riemannian manifold. It is easy to show that the torsion tensor $N - (dA) \otimes T = 0$, where N is the Nijenhuis tensor of f.

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2. Para-A-Einstein manifold. We define a para-A-Einstein manifold as a normal para contact Riemannian manifold whose Ricci tensor is given by [**1**]

$$\text{Ric}\,(X,Y) = a\,g(X,Y) + cA(X)A(Y) \tag{2.1}$$

where a and c are scalar functions. Obviously, we have

$$\text{Ric}\,(fX,Y) = \text{Ric}(XfY), \tag{2.2}$$

$$\text{Ric}\,(T,T) = a + c. \tag{2.3}$$

THEOREM 2.1. *The Riccian curvature of a para-A-Einstein manifold in the direction of T is equal to $-(n-1)$.*

Proof. From (1.3) and (1.4), we get

$$D_X T = fX. \tag{2.4}$$

From (1.5) and (2.4) we find the curvature tensor K satisfying

$$K(X,Y,T) = A(X)Y - A(Y)X. \tag{2.5}$$

Contracting (2.5), we get

$$\text{Ric}(Y,T) = -(n-1)A(Y). \tag{2.6}$$

Substituting T for Y in it we have the theorem.

THEOREM 2.2. *The functions a and c of the defining equation* (2.1) *are constants, provided tr. $f = 0$.*

Proof. Equation (2.3) and theorem (2.1) imply $a + c = 1 - n$. So we need only to show that a is constant. From (2.1), on contraction, we get $r = na + c$ which, on differentiation, yields

$$X_r = nX_a + X_c = (n-1)X_a, \tag{2.7}$$

where r is the scalar curvature. Again from (2.1) we have $R(X) = aX + cA(X)T$ which, on differentiation along Y, yields

$$(D_Y R)X = (Ya)X + (Yc)A(X)T + c(D_Y A)(X)T + cA(X)D_Y T.$$

The above equation assumes the form

$$(D_Y R)X = Ya + (Yc)A(X)T + c' f(X,Y)T + cA(X)fY$$

due to (2.4). Contracting in with respect to Y, we get $(\text{div}\,R)X = X_a + (T_c)A(X)$. Using the identity $(\text{div}\ R)X = X_r/2$ and (2.7), we get

$$(n-3)X_a = 2(T_c)A(X). \tag{2.8}$$

Putting $X = T$ in it, we get

$$(n-3)T_a = 2T_c = -2T_a$$

giving $T_a = 0$ and hence $T_c = 0$. Consequently (2.8) yields $X_a = 0$.

We now give a condition for a normal paracontact Riemannian manifold to be a para-A-Einstein manifold. With the help of (1.5) we can show for a normal paracontact Riemannian manifold that

$$\begin{aligned} K(X,Y,fZ) &= f(K(X,Y,Z)) + 2\{A(Y)'f(X,Z)T - A(X)'f(Y,Z)T \\ &\quad + A(Y)A(Z)fX - A(X)A(Z)fY\} -' f(X,Z)Y +' f(Y,Z)X \\ &\quad - g(Y,Z)fX + g(X,Z)fY. \end{aligned} \tag{2.9}$$

Contracting it with respect to X we find

$$\operatorname{Ric}(Y,fZ) = (C_1'\overline{K})(Y,Z) + (n-2)'f(Y,Z) + (C_1'f)\{2A(Y)Z(Z) - g(Y,Z)\} \tag{2.10}$$

where C_1' denotes contraction at the first slot and $\overline{K} \overset{\text{def}}{=} fK$.

$$(C_1'\overline{K})(Y,Z) = (C_1'\overline{K})(Z,Y), \tag{2.11}$$

From (2.10) and (2.11) it is obvious that

$$\operatorname{Ric}(X,fY) = \operatorname{Ric}(fX,Y). \tag{2.12}$$

THEOREM 2.3. *In order that a normal paracontact Riemannian manifold M_n may be a para-A-Einstein manifold it is necessary and sufficient that the symmetric tensors $C_1'\overline{K}$ and $'f$ should be linearly dependent.*

Proof. The theorem follows in consequence of equations (2.10), (2.1), (1.1)a, (2.6) and Theorem 2.2.

THEOREM 2.4. *In a para-A-Einstein manifold, the symmetric* $(0,2)$- *tensor $C_1'K$ is parallel along the vector field T.*

Proof. We have

$$(C_1'K)(Y,Z) = (a-n+2)'f(Y,Z) \tag{2.13}$$

due to (2.10) and (2.1). Differentiating it along T we have $(D_T C_1'\overline{K})(Y,Z) = 0$ due to (1.5).

THEOREM 2.5. *For a para-A-Einstein manifold, the Lie-derivatives of the Ricci tensor and $C_1'\overline{K}$ are given by*

$$L_T\operatorname{Ric} = (2a/(a-n+2))C_1'K, \tag{2.14}$$

$$L_T(C_1'\overline{K}) = 2(a-n+2)(g - A\otimes A). \tag{2.15}$$

Proof. It is easy to show for a normal paracontact Riemannian manifold that $L_TA = 0,\ L_Tf = 0,\ L_Tg = 2'f,\ L_T'f = 2(g - A\otimes A)$

From these relations and Lie-derivation of the equations (2.1) and (2.13) along T, the theorem follows.

3. Examples. *Example* (3.1) From [**3**] it is known that a neighborhood of each point of a manifold of constant curvature is a normal paracontact Einstein manifold which is therefore a trivial example of a para-A-Einstein manifold with $c = 0$.

Example (3.2). Next, we give an example of non-trivial para-A-Einstein manifold. Consider a $2(m+1)$–dimensional almost product and almost decomposable manifold $M_{2(m+1)}$ with structure tensor F such that the complementary distributions (having no common direction) may be of the same real dimension $m+1$. Suppose that $M_{2(m+1)}$ is of almost constant curvature [**5**]. Then its curvature tensor K is given by

$$(3.1)\quad '\tilde{K}(\tilde{X},\tilde{Y},\tilde{Z},\tilde{W}) = k[G(\tilde{X},\tilde{W})G(\tilde{Y},\tilde{Z}) - G(\tilde{X},\tilde{Z})G(\tilde{Y},\tilde{W}) \\ +'F(\tilde{X},\tilde{W})'F(\tilde{Y},\tilde{Z}) -' F(\tilde{X},\tilde{Z}) -' F(\tilde{X},\tilde{Z})'F(\tilde{Y},\tilde{W})]$$

where k is a constant, G is the metric tensor of $M_{2(m+1)}$ and $\tilde{X},\tilde{Y},\tilde{Z},\tilde{W}$ are arbitrary vector fields on it. Let M_{2m+1} be a hypersurface in $M_{2(m+1)}$ and M_{2m+1} be a normal paracontact Riemannian manifold with structure tensors f, T, A, g. Then it can be shown [**4**] that

$$(3.2(\text{a}))\qquad HX = -X + A(X)T,$$

$$(3.2(\text{b}))\qquad C'H = -2n,$$

where H is the second fundamental tensor of type (1.1) of the hypersurface. Since the dimension of the hypersurface is odd we can adapt an orthonormal frame $e_1, \dots, e_m,\ fe_1, \dots, fe_m, T$ on M_{2m+1}, with respect to which $C_1'f$ vanishes. Consequently div$T$ vanishes in view of (2.4) [**3**]. If B be the differential of the inclusion map $b : M_{2m+1} \to M_{2(m+1)}$, substituting BX, BY, BW for $\tilde{X},\tilde{Y},\tilde{Z},\tilde{W}$ in (3.1), we have

$$(3.3)\quad '\tilde{K}(BX,BY,BZ,BW) = [g(X,W)g(Y,Z) - g(X,Z)g(Y,W) \\ +' f(X,W)'f(Y,Z) -' f(X,Z)'f(Y,W)],$$

where we have used the transformation $FBX = BfX + A(X)N, N$ being the unit normal vector field to the hypersurface.

Using Gauss characteristic equation in (3.3) and contractin, we get

$$\text{Ric}(Y,Z) + h(HZ) - (C_1'H)h(Y,Z) = k[A(Y)A(Z) + (2m-1)g(Y,Z)].$$

Using (3.2)(a), (3.2)(b) frequently in the above equation, we find

$$\text{Ric}\,(Y,Z) - (2m-1)\{g(Y,Z) - A(Y)A(Z)\} = k\{(2m-1)g(Y,Z) + A(Y)A(Z)\}$$

which implies $\text{Ric}(Y,Z) = (k+1)(2m-1)g(Y,Z) + (k+1-2m)A(Y)A(Z)$, showing that the normal paracontact Riemannian hypersurface of almost product and almost decomposable manifold of almost constant curvature and whose complementary distributions have equal dimensions is a para-A-Einstein manifold.

It is notable that the scalar curvature of the enveloping manifold $M_{2(m+1)}$ is equal to $4n(n+1)k$.

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