

## FINSLER SPACE WITH RUND'S $h$ -CURVATURE TENSOR $K_{ljk}^i$ OF A SPECIAL FORM

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**Abstract.** We propose a special form of Rund's  $h$ -curvature tensor  $K_{ljk}^i$  and deal with some special Finsler spaces characterized by such a special form of  $K_{ljk}^i$ .

**1. Introduction.** Let us consider an  $n$ -dimensional Finsler space  $F_n (n \geq 3)$ , equipped with a metric tensor  $g_{ij}(x, y)$ , ( $y = x$ ) and a metric function  $L(x, y)$ . The  $h$ -covariant,  $v$ -covariant and  $o$ -covariant derivatives are denoted by  $|j$ ,  $\|j$  and  $\|j$  respectively.

A Finsler space is called  $h$ -isotropic [2] if Cartan's  $h$ -curvature tensor  $R_{hijk}$  is written in the form

$$(1.1) \quad R_{hijk} = R(g_{hj}g_{ik} - g_{hk}g_{ij}),$$

where  $R$  is non-zero scalar curvature. We have

$$(1.2) \quad R_{io{k}} = g_{ir}R_{ok}^r = RL^2h_{ik},$$

where  $h_{ik} = g_{ik} - l_i l_k$  is the angular metric tensor,  $R_{jk}^i$  is the  $(v)$   $h$ -torsion tensor and the suffix '0' means contraction with  $y^i$ .

A  $C$ -reducible Finsler space [3] is defined by

$$(1.3) \quad C_{ijk} = (C_i h_{jk} + C_j h_{ki} + C_k h_{ij}) / (n + 1)$$

where  $C_{ijk} \stackrel{\text{def}}{=} \dot{\partial}_k g_{ij} / 2 = g_{ij|k} / 2$ ,

$$(1.4) \quad C_i = C_{ijk} g^{jk}, \quad C_i y^i = 0,$$

A  $P$ -reducible Finsler space [6] is defined by

$$(1.5) \quad P_{ijk} = G_i h_{jk} + G_j h_{ki} + G_k h_{ij}$$

where

$$(1.6) \quad P_{ijk} = C_{ijk|0} \quad \text{and} \quad G_i = C_{i|0}/(n+1).$$

A Finsler space is called Landsberg if  $C_{ijk|o} = 0$ .

It is well known [1,5] that

$$(1.7) \quad \begin{aligned} \text{a) } & R_{jk||l}^i - K_{ljk}^i \{P_{jr}^i P_{kl}^r + P_{kl|j}^i\} = 0 \\ \text{b) } & K_{ljk}^i = R_{ljk}^i - C_{lr}^i R_{jk}^r, \quad \text{c) } R_{ojk}^i = R_{jk}^i, \\ \text{d) } & K_{ljk}^i + K_{jkl}^i + K_{kij}^i = 0, \\ \text{e) } & H_{ljk}^i = R_{jk||l}^i, \quad \text{f) } H_{ljk}^i + H_{jkl}^i + H_{kij}^i = 0, \\ \text{g) } & K_{ljk}^i = -K_{lkj}^i, \quad K_{lijk} = g_{ir} K_{ljk}^r \\ \text{h) } & H_{ljk}^i = K_{ljk}^i + U_{(jk)} \{C_{jt|o|k}^i + C_{kr|o}^i C_{jt|o}^r\}, \end{aligned}$$

where  $U_{(jk)} \{A_j B_k\} = A_j B_k - A_k B_j$ . The angular metric tensors  $h_{ij}$  and  $h_j^i (= g^{ik} h_{kj})$  have the following covariant differentiations

$$(1.8) \quad \begin{aligned} \text{a) } & h_{ij|k} = 0, \quad h_{ij}^i|k = 0 \\ \text{b) } & h_{ij||k} = 2C_{ijk} - L^{-2}(y_i h_{jk} + y_j h_{ik}) \\ \text{c) } & h_{j||k}^i = -L^{-2}(y_j h_k^i + y^i h_{jk}). \end{aligned}$$

We shall use the following lemma proved by Matsumoto [4].

LEMMA. *If the equation  $v_{hi} h_{jk} + v_{ij} h_{hk} + v_{jh} h_{ik} = 0$  holds in  $F_n$ , then we have (1)  $v_{ij} = 0$ , ( $n \geq 4$ ) and (2)  $v_{ij} = v(m_i n_j - m_j n_i)$ , with reference to Moor frame  $(l^i, m^i, n^i)$  where  $v$  is a scalar.*

**2. Special form of Rund's h-curvature tensor  $K_{ljk}^i$ .** Let  $F_n$  be a Finsler space with Rund's  $h$ -curvature tensor  $K_{ljk}^i$  of the form

$$(2.1) \quad K_{ljk}^i = U_{(jk)} \{A_{jk} h_l^i + B_{jl} h_k^i + D_k^i h_{ji}\}$$

where  $A_{jk}$ ,  $B_{jl}$  and  $D_k^i$  are Finsler tensor fields.

THEOREM 2.1. *A  $P$ -reducible Finsler space of non zero scalar curvature has the special form (2.1) of Rund's  $h$ -curvature tensor  $K_{ljk}^i$ .*

*Proof.* From (1.7 a)

$$(2.2) \quad K_{ljk}^i = R_{jk||l}^i + U_{(jk)} \{P_{jr}^i P_{kl}^r + P_{kl|j}^i\}$$

since  $R_{jk}^i = (R_{ok||j}^i - R_{oj||k}^i)/3$ .

From (1.2) and (1.8 c) we get

$$(2.3) \quad R_{jk}^i = a_j h_k^i - a_k h_j^i$$

where  $a_j = (R_{||j} L^2 + 3R y_j)/3$ .

From (1.5) and (2.3) we have

$$(2.4) \quad P_{kl|j}^i = G_{k|j}h_l^i + G_{l|j}h_k^i + G_{|j}^i h_{kl}$$

(2.5)

$$R_{jk||l}^i = L^{-2}(a_k y_j - a_j y_k)h_l^i + a_{j||l}h_k^i - a_{k||l}h_j^i + L^{-2}a_k y^i h_{jl} - L^{-2}a_j y^i h_{kl}$$

respectively. Using (1.5), (2.4) and (2.5) in (2.1), we get

(2.6)

$$K_{ljk}^i = \{(L^{-2}a_k y_j + G_{k|j}) - (L^{-2}a_j y_k + G_{j|k})\}h_l^i + (a_{j||l} + G_{l|j} - G_j G_l)h_k^i \\ - (a_{k||l} + G_{l|k} - G_k G_l)h_j^i + (L^{-2}a_k y^i - M_k^i)h_{jl} - (L^{-2}a_j y^i - M_j^i)h_{kl},$$

where  $M_k^i = G_k G^i + G_r G^r h_k^i + G_{|k}^i$ .

The equation (2.6) reduces to the form (2.1) with

$$A_{jk} = L^{-2}a_k y_j + G_{k|j}, \quad B_{jl} = a_{j||l} + G_{l|j} - G_j G_l, \quad D_k^i = L^{-2}a_k y^i - M_k^i.$$

**COROLLARY 1.** *A Landsberg space of non-zero scalar curvature has the special form (2.1) of Rund's  $h$ -curvature tensor.*

*Proof.* For a Landsberg space  $P_{ijk} = 0$ ; hence from (1.7 a) and (2.5) we have (2.1) with

$$A_{jk} = L^{-2}a_k y_j, \quad B_{jl} = a_{j||l}, \quad D_k^i = L^{-2}a_k y^i.$$

**THEOREM 2.2.** *An  $h$ -isotropic and  $C$ -reducible Finsler space  $G_n$  has the special form (2.1) of Rund's  $h$ -curvature tensor  $K_{ljk}^i$ .*

*Proof.* If the space is  $h$ -isotropic and  $C$ -reducible then by (1.1) and (1.3) the equation (1.7 b) takes the form

(2.7)

$$K_{ljk}^i = (C_j y_k - C_k y_j)h_l^i / (n+1) + R(h_{lj} + l_i l_j - C_l y_j / (n+1))h_k^i \\ - R(h_{lk} + l_l l_k - C_l y_k / (n+1))h_j^i + R(l^i l_k + C^i y_k / (n+1))h_{lj} \\ - R(l^i l_j + C^i y_j / (n+1))h_{lk},$$

which reduces to the form (2.1) with

$$A_{jk} = R(C_j y_k) / (n+1), \quad B_{jl} = R(h_{ij} + l_i l_j - C_l y_j / (n+1)), \quad D_k^i = R(l^i l_k + C^i y_k / (n+1)).$$

**THEOREM 2.3.** *If the Rund's  $h$ -curvature tensor has the special form (2.1), then  $F_n$  is a space of scalar curvature  $B_{oo}L^{-2}$ .*

*Proof.* From (1.7b) and (2.1) we have

$$(2.8) \quad R_{ljk}^i - C_{ir}^i R_{jk}^r = U_{(jk)} \{A_{jk} h_l^i + B_{jl} h_k^i + D_k^i h_{jl}\}.$$

Transvecting (2.8) by  $y^l$  we get

$$(2.9) \quad R_{jk}^i = B_{j\circ} h_k^i - B_{k\circ} h_j^i.$$

Tarnsvecting (2.9) by  $y^j$  we get

$$(2.10) \quad R^i_{ok} = B_{oo}h^i_k.$$

Comparing (2.10) with (1.2) we get  $R = B_{oo}L^{-2}$ .

**THEOREM 2.4.** *A C-reducible Finsler space satisfying (2.1) has Cartan's h-curvature tensor  $R^i_{ljk}$  of the form*

$$(2.11) \quad R^i_{ljk} = U_{(jk)}\{Q_{jk}h^i_l + N_{jl}h^i_k + E^i_k h_{jl}\}.$$

*Proof.* Using (1.3), (2.1) and (2.9) in (1.7 b) we get

$$(2.12) \quad R^i_{ljk} = U_{(jk)}\{(A_{(jk)} + C_j B_{ko}/(n+1))h^i_l + (B_{jl} + C_l B_{jo}/n+1)h^i_k + (D^i_k - C^i B_{ko}/(n+1))h_{jl}\}$$

which reduces to the form (2.11) with

$$Q_{jk} = A_{jk} + C_j B_{ko}/(n+1), \quad N_{jl} = B_{jl} + C_l B_{jo}/(n+1), \quad E^i_k = D^i_k - C^i B_{ko}/(n+1)$$

**THEOREM 2.5.** *If Rund's h-curvature tensor  $K^i_{ljk}$  is of the form (2.1), then we have*

$$(2.13) \quad \begin{aligned} (a) \quad & A_{jk} - A_{kj} = B_{jk} - B_{kj}, \quad b) \quad B_{jo} = (R_{||j}L^2 + 3Ry_j)/3, \\ (c) \quad & (B_{jo||k} - B_{ko||j})g^{jk} = 0. \end{aligned}$$

*Proof.* Using the Bianchi identity (1.7d) in (2.1) we get

$$(A_{jk} - A_{kj} + B_{kj} - B_{jk})h^i_l + (A_{kl} - A_{lk} + B_{lk} - B_{kl})h^i_j + (A_{lj} - A_{jl} + B_{jl} - B_{lj})h^i_l = 0$$

which due to lemma yields (2.13a). Again from (2.3) and (2.10) we get (2.13b).

Now differentiating (2.9) 0-covariantly and using (1.7 e) and (1.8 c) we get

$$H^i_{ljk} = R^i_{jkl||l} = L^{-2}(B_{ko}y_j - B_{jo}y_k)h^i_j + U_{(jk)}\{B_{jo||l}h^i_k + L^{-2}B_{ko}y^i h_{jl}\},$$

which due to (1.7f) yields  $U_{jk}h^i_l + U_{kl}h^i_j + U_{lj}h^i_k = 0$ , where  $U_{jk} = (B_{ko}y_j - B_{jo}y_k)L^{-2} + B_{ko||j} - B_{jo||k}$ . Applying the lemma and transvecting by  $g^{jk}$  we get  $y_k$  (2.13 c).

**THEOREM 2.6.** *A Finsler space with Rund's h-curvature of the form (2.1) is a Riemannian space of constant curvature  $B_{oo}L^{-2}$ .*

*Proof.* Using Theorem 2.3 and Corollary 1 and [5, Th. 30.6] we obtain the theorem.

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