

ON THE DEGREE OF L_1 -APPROXIMATION BY MODIFIED BERNSTEIN POLYNOMIALS

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Abstract. Recently many researchers like Bojanić & Shisha, and A. Grundmann have obtained the degree of L_1 approximation to integrable functions by modified Bernstein polynomials. The object of the present note is to improve their results.

1. Introduction. The modified Bernstein polynomials

$$(1.1) \quad (P_n f)(x) = (n+1) \sum_{k=0}^n p_{n,k}(x) \int_{I_k} f(t) dt$$

Where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $I_k = [k/(n+1), (k+1)/(n+1)]$ were introduced by Kantorovič to approximate Lebesgue integrable functions in L_1 norm. Using a weight function $W(x) = \sqrt{x(1-x)}$, Bojanić and Shisha have recently obtained the following rate of convergence in terms of integral modulus of continuity,

$$w(f, \delta)_{L_1} = \sup_{|h| \leq \delta} \int_0^1 |f(x+h) - f(x)| dx$$

by operators (1.1).

THEOREM 1. *Let f be a Lebesgue integrable function in $[0, 1]$. Then for $n \geq 2$,*

$$(1.2) \quad \int_0^1 W(x) | (P_{n-1} f)(x) - f(x) | dx \leq 2\pi/3 \cdot w(f; 1/\sqrt{n})_{L_1}.$$

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Later on, A. Grundmann [2] has given the following result.

THEOREM 2. *Let f be a Lebesgue integrable function in $[0, 1]$. Then for $n \geq 2$,*

$$(1.3) \quad \int_0^1 |(P_{n-1})f(x) - f(x)| dx \leq 4w(f; 1/\sqrt{n})_{L_1}.$$

The object of the present note [4] is to improve the results (1.2) and (1.3) because not much is known about the degree of L_1 approximation to integrable functions by modified Bernstein polynomials.

2. We need the following lemmas to prove our theorem.

LEMMA 2.1. *Let $f_\delta(y) = 1/\delta \cdot \int_0^\delta f(y+t) dt$ ($\delta > 0$), then*

$$\|f_\delta^{(1)}(y)\|_{L_1} \leq 1/\delta \cdot w(f; \delta)_{L_1} \quad \text{and} \quad \|f - f_\delta\|_{L_1} \leq w(f; \delta)_{L_1}.$$

Proof. The proof follows from [2].

LEMMA 2.2. *For $0 \leq y \leq 1$ and $0 \leq r \leq n-1$, one has*

$$(1-r/n)p_{n,r}(y) \leq \sqrt{3}/8\sqrt{n}.$$

Proof. We know that

$$\begin{aligned} (n-r)/\sqrt{n} \cdot p_{n,r}(y) &\leq \binom{n-1}{r} (r+1)^{r+1} (n-r)^{n-r} \sqrt{n} (n+1)^{-n-1}, \quad (0 \leq r \leq n-1) \\ &\leq \binom{n-1}{[n/2]} ([n/2]+1)^{[n/2]+1} (n-[n/2])^{n-[n/2]} \sqrt{n} (n+1)^{-(n+1)} = U_n \quad (\text{Say}) \end{aligned}$$

where $[n/2]$ denotes the largest integer not exceeding $n/2$.

Case 1. When $n = 2m + 1$, ($m \geq 1$). Then

$$\begin{aligned} U_n &= \binom{2m}{m} (m+1)^{2m+2} \sqrt{2m+1} / (2m+2)^{2m+2} = \binom{2m}{m} \sqrt{2m+1} / 2^{2m+2} \\ &= \frac{1 \cdot 3 \cdot 5 \cdots (2m-1)}{2 \cdot 4 \cdot 6 \cdots 2m} \cdot \frac{\sqrt{2m+1}}{4} \\ &= 1/4 \sqrt{(1-1/4)(1-1/4 \cdot 2^2)(1-1/4 \cdot m^2)} \leq 1/4 \cdot \sqrt{3/4} = \sqrt{3}/8 \quad (m \geq 1). \end{aligned}$$

Case 2. When $n = 2m$, ($m \geq 1$). Then

$$\begin{aligned}
 U_n &= (2m-1)! \sqrt{2m} m^m (m+1)^{m+1} / (m-1)! m! (1+2m)^{2m+1} \\
 &= \frac{1 \cdot 3 \cdot \dots \cdot (2m-1)}{2 \cdot 4 \cdot \dots \cdot 2m} \cdot \frac{\sqrt{2m+1}}{4} \cdot \frac{m^{m+1/2} (m+1)^{m+1}}{(1/2+m)^{2m+3/2}}.
 \end{aligned}$$

Since $m^{2m+1} (1+m)^{2m+2} \leq (m+1/2)^{4m+3}$ for $m \geq 1$, we have $U_n = U_{2m+1} \leq \sqrt{3/8}$ by case 1.

LEMMA 2.3. Let $I_1 = \int_0^1 | (P_{n-1} f_y)(x) - f_y(x) | dx$. Then for $n \geq 2$, $0 \leq y \leq 1$, $I_1 \leq 23/20 \sqrt{n}$, where

$$f_y(x) = \begin{cases} 1 & \text{when } x \leq y \\ 0 & \text{otherwise} \end{cases}$$

Proof. We get clearly $I_1 = I_2 + I_3$ (say), where

$$\begin{aligned}
 I_2 &= \int_0^y | P_{n-1} f_y(x)(x) - f_y(x) | dx \\
 &= \int_0^y \left| \sum_{k=0}^{n-1} p_{n-1,k}(x) - n \sum_{k=0}^{n-1} p_{n-1,k}(x) \int_{k/n}^{k+1/n} f_y(du) \right| dx \\
 &= \int_0^y \left| \sum_{k=0}^{n-1} p_{n-1,k}(x) - \sum_{k=0}^{[ny]} p_{n-1,k}(x) \right| dx = \sum_{k=[ny]+1}^{n-1} \int_0^y p_{n-1,k}(x) dx, \\
 I_3 &= \int_y^1 | P_{n-1} f_y(x)(x) - f_y(x) | dx = \sum_{k=0}^{[ny]} \int_y^1 p_{n-1,k}(x) dx \\
 &= \sum_{k=0}^{[ny]} \left\{ \int_0^1 p_{n-1,k}(x) dx - \int_0^y p_{n-1,k}(x) dx \right\} \\
 &= \int_0^y \left\{ 1 - \sum_{k=0}^{[ny]} p_{n-1,k}(x) \right\} dx + ([ny] + 1 - ny)/n \\
 &\leq \sum_{k=[ny]+1}^{n-1} \int_0^y p_{n-1,k}(x) dx + 1/n
 \end{aligned}$$

Using $\int_0^y p_{n-1,k}(x) dx = 1/n \cdot \sum_{l=k+1}^n p_{n,l}(y)$ we get easily that

$$\begin{aligned}
I_1 &\leq 2/n \cdot \sum_{k=[ny]+1}^{n-1} \sum_{l=k-1}^n p_{n,l}(y) + 1/n \\
&= 2/n \cdot \sum_{k=[ny]+1}^n \{k - [ny] - 1\} p_{n,k}(y) + 1/n \leq 2 \sum_{k=[ny]+1}^n (k/n - y) p_{n,k}(y) + 1/n \\
&= 2 \binom{n-1}{[ny]} y^{[ny]+1} (1-y)^{n-[ny]} + 1/n \leq \sqrt{3}/4\sqrt{n} + 1/n \quad (\text{Using Lemma 2.2}) \\
&\leq 1/\sqrt{n} \cdot (\sqrt{3}/4 + \sqrt{2}/2) \leq 23/20\sqrt{n}, \quad (n \geq 2)
\end{aligned}$$

This completes the proof

We prove the following result.

THEOREM 3. *If f is a Lebesgue integrable function in $[0, 1]$ then for $x \geq 2$,*

$$(2.4) \quad \int_0^1 |(P_{n-1}f)(x) - f(x)| dx \leq 63/20 \cdot w(f; 1/\sqrt{n})_{L_1}.$$

Proof. Using $\|P_{n-1}\|_{L_1} = 1$, it can be obtained that

$$(2.5) \quad \begin{aligned} \|(P_{n-1}(f) - f)\|_{L_1} &\leq \|P_{n-1}(f - f_\delta)\|_{L_1} + \|P_{n-1}(f_\delta) - (f_\delta)\|_{L_1} + \|(f_\delta - f)\|_{L_1} \\ &\leq 2\|(f_\delta - f)\|_{L_1} + \|(P_{n-1}(f_\delta) - (f_\delta))\|_{L_1}. \end{aligned}$$

We can show similarly to what is done in [3] that

$$\|(P_{n-1}(f_\delta) - (f_\delta))\|_{L_1} \leq \int_0^1 |f_\delta^{(1)}(y) R_n(y)| dy$$

$$\text{where} \quad R_n(y) = \int_0^1 |(P_{n-1}f_x(y)(x) - f_x(y)| dx.$$

Clearly $R_n(y) = \int_0^1 |(P_{n-1}f_y(x)(x) - f_y(x)| dx$, [because $f_x(y) = 1 - f_y(x)$ everywhere, except for the point $x = y$ and $(P_{n-1}1)(x) = 1$]. So

$$(2.6) \quad R_n(y) = I_1 \leq 23/20\sqrt{n}, \quad n \geq 2, \quad 0 \leq y \leq 1, \quad (\text{Using Lemma 2.3}).$$

Again using Lemma 2.1 and the result (2.6) in (2.5), we get

$$\begin{aligned}
\|(P_{n-1}(f) - f)\|_{L_1} &\leq 2w(f; \delta)_{L_1} + 23/20\sqrt{n} \cdot \int_0^1 |f_\delta^{(1)}(y)| dy \\
&\leq 2w(f, \delta)_{L_1} + 23/20\sqrt{n} \cdot \int_0^1 |f_\delta^{(1)}y| dy \leq (2 + 23/20\delta\sqrt{n})w(f, \delta)_{L_1}.
\end{aligned}$$

Finally choosing $\delta = 1/\sqrt{n}$, we get the required result (2.4). This completes the proof.

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