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THE a–TEMPERED DERIVATIVE AND SOME SPACES OF EXPONENTIAL DISTRIBUTIONS

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In this paper we introduce the a-tempered integral and the a-tempered derivative for which almost all results from [**5**] can be simply transferred. Using the special sequences of a-tempered integrals and and a-tempered derivatives, from the point of view of sequential approach, we characterize some subspaces of D'. These spaces are of $K\{M_p\}$-type ([**2**]) for the special sequences $\{M_p\}$.

The a-tempered integral and a-tempered derivative

We are going to define the a-tempered derivative and the a-tempered integral similarly as in [**1**], p. 161.

Let $x \to a(x)$, $x \in R$, be an infinitely differentiable function. If $f \in D'$ and $k \in N$, the a-tempered derivative of order k is defined by

$$(1)\qquad D_a f = \exp(-a(x))(\exp(a(x))f(x))'; \quad D_a^0 f = f; \quad D_a^k f = D_a(D_a^{k-1} f)$$

It is clear that

$$(2)\qquad D_a f = a'f + f'; \quad gD_a f = D_a(fg) - g'f$$

where $g(x) \in C^\infty$.

The a-tempered integral of a function $G(x) \in L^1_{loc}(R)$ of order $k \in N$ is defined by

$$(3)\qquad S_a G = \exp(-a(x)) \int_0^x \exp(a(t))G(t)dt; \quad S_a^0 G = G; \quad S_a^k G = S_a(S_a^{k-1} G).$$

If $G \in L^1_{loc}$ then $D_a^k S_a^k G = G$, but the converse does not hod.

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The operators S_a^k and D_a^k are linear for any $k \in N$. It is easy to verify that

$$S_a^k G = \exp(-a(x)) \int_0^x \frac{(x-t)^{k-1}}{\Gamma(k)} G(t) \exp(a(t)) dt, \quad k \in N.$$

If we define for $\alpha \geq 0$

$$S_a^k G = \exp(-a(x)) \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \exp(a(t)) G(t) dt,$$

we may prove as in ([**5**]) that $S_a^{\alpha+\beta}\, G = S_a^\alpha S_a^\beta G,\ \alpha \geq 0,\ \beta \geq 0.$

In [**5**] the so-called rapidly decreasing functions in zero (RDZ) are defined. Let us repeat the definition:

The function f is RDZ iff for every $r \in N$ there exists $M_r > 0$ such that $\mid f(x) \mid \leq M_r \mid x \mid^r$ for $\mid x \mid \leq 1$.

The set of RDZ functions is a linear space.

Since all results from [**5**] can be easily transfered for the a-tempered derivative and integral we shall only state these facts here.

If f is RDZ function then for any $\alpha \geq 0$ $S_a^\alpha f$ iz RDZ function and $S_a^\alpha f$ has the value 0 at $x = 0$.

We say that $f \in D'$ is a-rapidly decreasing distribution (a-RDZ) if f is of the form $f = D_a^k F$ for some $k \in N$ and some RDZ function F.

The set of such distributions forms a subspace of the space D' and evert such distribution has the value 0 at $x = 0$.

For any a-RDZ distribution f there exists a unique a-RDZ distribution g such that $f = D_a g$. This result is used in defining the a-tempered derivative of order $\alpha \geq 0$ of a-RDZ distributions as $D_a^\alpha f = D_a^{p+l}\, S_a^{p-\alpha} f$ where $f = D_a^l F$ for some continuous RDZ functions, $l \in N$ and p is an integer such that $0 \leq p-1 < \alpha \leq p$. The operator $D_a^\alpha,\ \alpha \geq 0$, is linear in the space of a-RDZ distributions and for any $\alpha \geq 0,\ \ \beta \geq 0\ \ D_a^\alpha D_a^\beta f = D_a^{\alpha+\beta} f$ where f is an a-RDZ distribution.

In our further observations for the first derivative of the function $a(x)$ we shall suppose that there exist $C > 0$ and $m > 0$ such that $a'(x) \geq m$ for $x > C$ and $a'(x) \leq -m$ for $x < -C$.

Let us prove some properties of the operator S_a.

LEMMA 1. *(i) If* $F \in L^2$ *then* $S_a F \in L_2$*;* *(ii) If* (F_n) *is sequence from* L^2*, and* $F_n \overset{2}{\to} F$ *then* $S_a F_n \overset{2}{\to} S_a F$*;* *(iii) If* $F \in L^1_{loc}$ *then* $\mid \exp(-a(x)) S_a F \mid \leq S_a(\exp(-a(x)) \mid F \mid)$.

Proof. (i) We shall use the idea of the proof of Lemma 7.4.2 from [**1**]. As some tehnical changes are needed, we shall give the complete proof of this assertion.

Let us denote

$$I_B = \int_0^B \exp(-2a(x))M^2(x)dx$$

where

$$M(x) = \int_0^x \exp(a(t)) \mid F(t) \mid dt \quad \text{and} \quad B > 0.$$

As we supposed there exists $C \geq 0$ such that for $x \geq C$ $a'(x) > m$.
Since

$$I_B = \int_0^C + \int_C^B,$$

first we shall estimate $\int_0^C$ and after that, $\int_C^B$.

$$\int_0^C \exp(-2a(x))\Big(\int_0^x \exp(a(t)) \mid F(t) \mid dt\Big)^2 dx \leq K_0 \int_0^C \Big(\int_0^x \exp(a(t)) \mid F(t) \mid dt\Big)^2 dx \leq$$

$$K_0 \int_0^C \Bigg(\Big(\int_0^x \exp(2a(t)dt\Big)\Big(\int_0^x \mid F(t) \mid^2 dt\Big)\Bigg) dx.$$

So if $A = \int_{-\infty}^{\infty} \mid F(t) \mid^2 dt$, for a suitable K_1 we obtain

$$\Big| \int_0^C \Big| \leq K_1 A. \tag{4}$$

Since $a'(x) > m$ for $x \in [C, \infty)$, there exists $\alpha > 0$ such that $2\alpha a'(x) \geq 1$ (for $x \in [C, \infty)$). By the partial integration we obtain

$$\int_C^B \exp(-2a(x))M^2(x)dx \leq \alpha \int_C^B 2a'(x) \exp(-2a(x))M^2(x)dx =$$

$$= \alpha \int_C^B (-\exp(-2a(x)))' M^2(x)dx \leq 2\alpha \int_C^B \exp(-2a(x))M(x) \exp(a(x)) \mid F(x) \mid dx \leq$$

$$\leq 2\alpha a\Big(\int_C^B \exp(-2a(x))M^2(x)dx\Big)^{1/2}\Big(\int_0^B \mid F(x) \mid^2 dx\Big)^{1/2}.$$

From that it follows that for a suitable K_2

$$\left| \int_C^B \right| \leq K_2\sqrt{I_B A}. \tag{5}$$

From (4) and (5) we obtain $I_B \leq K_1 A + K_2\sqrt{I_B A}$.

If $I_B \geq K_1 A$ than $(I_B - K_1 A)^2 \leq K_2^2\ I_B A$ and $I_B \leq (2K_1 + K_2^2)A$.

In any case there exists $K_3 > 0$ such that $I_B \leq K_3 A$.

Similarly, we can prove that $\left| \int_0^{-B} \exp(-2a(x))M^2(x)dx \right| \leq K_4 A$ and so we obtain the assertion.

(ii) If in the proof of the part (i) we put

$$M_n(x) = \int_0^x \exp(a(t)) \mid F_n(t) - F(t) \mid dt \ \text{ and } \ A_n = \int_{-\infty}^{\infty} \mid F_n(t) - F(t) \mid^2 dt$$

then the assertion follows from the inequality

$$\int_{-\infty}^{\infty} \exp(-2a(x))M_n^2(x)dx \leq KA_n, \quad \text{where } K = \max(K_3, K_4).$$

(iii) is simple.

Remark. If $(a(x) = o(x)$ when $x \to \infty$, similarly as in [**1**] we may prove that $S_a(1) \in L^2$.

Some spaces of exponential distribution

Let $(\tilde{m}_p(x)),\ p \in N,\ 0 \leq x < \infty$, be a sequence of nondecreasing continuous functions such that for every $p, \tilde{m}_p(o) = 0; \tilde{m}_p(x) \to \infty$ as $x \to \infty; \tilde{m}_1(x) \leq \tilde{m}_2(x) \leq \cdots$. We define

$$m_p(x) = \int_0^{|x|} \tilde{m}_p(t)dt, \ \ x \in R, \ \ p \in N. \tag{6}$$

This implies that for every $p \in N$ the functions $m_p(x)$ are convex (this implies that if $x \cdot y \geq 0$ then $m_p(x) + m_p(y) \leq m_p(x+y)$) and increase to infinity faster than any linear function when $\mid \dot{x} \mid \to \infty$. We suppose that the following condition is satisfied

(A) For every $p \in N$ there exists $x_p > 0$ and $p' \in N$, such that $m_p(px) \leq m'_p(x)$ for $\mid x \mid \geq x_p$.

In [**6**] we have proved that (A) implies the so-called nuclearity condition for the sequence $(\exp(m_p(x)))$:

(N) For every p there exists $p' \in N$, such that $\exp(m_p(x)-m_{p'}(x))$ is a summable function on R and $\exp(m_p(x)-m_{p'}(x)) \to 0$ when $| x |\to \infty$.

Also, we suppose that for the elements of the sequence $(\exp(m_p(x)))$ the following condition holds.

(E) For every $p \in N$ and every $k \in N$ there are $\varepsilon_k \in (0,1)$, $C_k > 0$ and $\bar{x}_p > 0$ such that $m_p^k(x) < C_k \exp(m_p((1-\varepsilon_k)(x)))$ if $| x |\geq \bar{x}_p$.

We shall use the sequence $(n_p(x))$ constructed in the following way:

Let $\omega(x)$ be a smooth positive function on R such that $\operatorname{supp}\omega \in [0;1]$ and $\int_R \omega(x)dx = 1$. For $| x |> 1$ we put $n_p(x) = \bar{n}_p(| x |)$, where $\bar{n}_p(x) = (m_p(t) * \omega(t))(x)$, $x > 1$. For $| x |\leq 1$ we define $n_p(x)$ to be smooth non-decreasing, positive and $n_p(x) \leq n_{p+1}(x)$, $p \in N$.

It is easy to verify that for $x \geq 1$

$$m_p(x-1) \leq n_p(x) \leq m_p(x). \tag{7}$$

Every function $n_p(x)$, $p \in N$, satisfies conditions as the function $a(x)$ from the first part of the paper. So using these functions we define the sequence of n_p-tempered integrals (S_{np}) and the sequence of n_p-tempered derivatives (D_{np}).

Let us define a subset of D' in the following way: A distribution f is in H' iff there exist $p \in N$, $k \in N_0$ and a locally integrable function F for which $F(x)\exp(-n_p(x)) \in L^2$, such that

$$f = D_{n_p}^k F. \tag{8}$$

We are going to show that H' is a subspace of D' identical to the space $H'\{\exp m_p(x))\}$ which we have introduced in | 6 |.

Theorem 1. *A distribution f is in H' iff there exist $p \in N$, $m \in N$ and a bounded continuous function $F(x)$ such that*

$$f(x) = (F(x)\exp(m_p(x)))^{(m)}. \tag{9}$$

Proof. If (8) holds for the corresponding p, and F, then $f = D_{np}^{k+1}F_1$ where $F_1 = S_{np}F$ is a continuous function. From Lemma 1 (iii) and (i) it follows

$$\exp(-n_p(x))S_{n_p} | F(x) |\leq S_{n_p}(\exp(-n_p(x)) | F(x) |) \in L^2.$$

Applying (2) we obtain

$$D_{n_p}^{k+1}F_1 = \sum_{l=0}^{k+1} c_l(N_l(x)F_1(x))^{(l)} \tag{10}$$

where c_l are the corresponding constants, $N_l(x)$ are the products of the members of the form $(n_p^{(r)})^s$, $r \leq k+1$, $s \leq k+1$, with the corresponding r and s which depend on l.

From the construction of $n_p(x)$ and condition (E) it follows that for sufficiently large $| x |$ and suitable C and C_1

$$\sup_{l \leq k+1} | N_l(x) | \leq C \mid m_p^{k+1}(x) | \leq C_1 \exp(m_p((1-\varepsilon_{k+1})x)) \tag{11}$$

since $| n_p^{(r)}(x) | \leq M_r \mid m_p(x) |$ for some constant M_r.

For $F_1(x)$ the folowing estimate holds.

$$| F_1(x) | \leq S_{n_p} \mid F(x) | = \exp(-n_p(x)) \int_0^x \exp(2n_p(t)) \mid F(t) \mid \exp(-n_p(t))dt \leq$$

$$\leq \big(\exp(-2n_p(x)) \int_0^x \exp(4n_p(t))dt\big)^{1/2} \cdot \big(\int_R \mid F(t) \mid^2 \exp(-2n_p(t))dt\big)^{1/2}.$$

In fact, for the corresponding $C > 0$ we have

$$| F_1(x) | \leq C\sqrt{x} \exp(-n_p(x)). \tag{12}$$

In order to simplify notations, from this point up to the end of the paper we shall put $n_{po}, m_{po}, \ldots$, instead of $n_{po}, m_{po}, \ldots$.

From (7), condition (A) and convexity of the function $m_p(x)$ for some $\varepsilon > 0$ for which $\varepsilon_{k+1} - \varepsilon > 0$ holds, it follows

$$\begin{aligned} &- n_{po}(x) \leq -m_{po}(x-1) \leq -m_{po}((1-\varepsilon_{k+1})x + \varepsilon x) \leq \\ &\leq -m_{po}((1-\varepsilon_{k+1})x) - m_{po}(\varepsilon x) \end{aligned} \tag{13}$$

for sufficiently large $| x |$.

Using (11), (12) and (13) if $p_0 > p$ we obtain (for some new $C > 0$)

$$\exp(n_{p0}(x)) \exp(-n_{p0}(x)) \mid F_1(x) \mid\mid N_l(x) \mid \leq C\sqrt{x} \exp(-m_{p0}(\varepsilon x) + n_p(x) + n_{p0}(x))$$

Since $m_p(x)$ increases to infinity faster than any linear function, from (A) it follows that for a given p there exists $p_0 > p$ such that $\sqrt{x} \exp(-m_{p0}(\varepsilon x) + n_p(x))$ is bounded on R. It means that

$$D_{n_p}^{k+1} F_1 = \sum_{l \leq k+1} (\exp(n_{p0}(x)) \overline{F}_l(x))^{(l)}$$

where $\overline{F}(x)$ are the corresponding bounded continuous functions.

By the partial integration we obtain

$$\int_0^x \exp(n_{p0}(t)) \overline{F}_l(t) dt = \exp(n_{p0}(u)) \int_0^u \overline{F}_l(t) dt \Big|_0^x -$$

$$- \int_0^x \big(n'_{p6}(u) \exp(n_{p0}(u)) \int_0^u \overline{F}_l(t) dt\big) du.$$

From (E) it follows that

$$\int_0^x \exp(n_{p0}(t))\overline{F}_l(t)dt = \exp(n_{p1})\overline{\overline{F}}_l$$

where $\overline{\overline{F}}_l(x)$ is the corresponding bounded continuous function and p_1 is the corresponding integer greater than p_0. Using the preceding argument sufficiently many times we obtain that for some new p and new F (9) holds.

Let us suppose that (9) holds. From (7) and (A) it follows that there exists $p_1 > p$ such that $m_p(x) \leq n_p(x+1) \leq n_p(2x) \leq n_p,\ (x)$ holds for sufficiently large $| x |$.

Since $F(x)$ is bounded, from (E) it follows that there exists $p_0 > p_1$ such that

$$F(x)\exp(-n_{p0}(x) + m_p(x))(n_{p0}^{(l)})^r \in L^2$$

for any $l \leq m,\ \ r \leq m$. If we put

$$\tilde{F}(x) = F(x)\exp(-n_{p0}(x) + m_p(x)) \ \text{ than } \ F(x) = (\tilde{F}(x)\exp(n_{p0}(x))^{(m)}.$$

After using the Leibniz formula and (2) we obtain that $f(x)$ is a linear combination of expressions of the form

$$D_{n_p0}^j(S_{n_p0}^r(n_{p0}^{(l)}(x))^s\tilde{\tilde{F}}),\ r,\ l,\ s \leq j \leq m, \ \text{ where } \ \tilde{\tilde{F}} = \tilde{F}\exp(n_{p0}(x)).$$

From the fact $(n_{p0}^{(l)}(x))^s\tilde{F} \in L^2$ and Lemma 1 (iii) it follows that this expression can be represented in the form of

$$D_{n_p0}^j\tilde{\tilde{\tilde{F}}}_j \quad \text{where} \quad \exp(-n_{p0})F_j \in L_2$$

From the linearity of the operator D_{n_p0} and from the identity

$$D_{n_p0}^m(S_{n_p0}^{m-j}\tilde{F}_j) = D_{n_p0}^j\tilde{F}_j. \tag{14}$$

it follows that $f \in H'$.

THEOREM 2. *The set H' is a linear spase.*

Proof. We have only to prove that if

$$f_1 = D_{n_p3}^{r_1}\tilde{F}_1 \quad \text{and} \quad f_2 = D_{n_p4}^{r_2}\tilde{F}_2$$

then $f_1 + f_2 \in H'$.

From the preceding theorem it follows that

$$f_1(x) = (\exp(m_{p1}(x))F_1(x))^{(m1)} \ \text{ and } \ f_2(x) = \exp(m_{p2}(x))F_2(x))^{(m2)}$$

for the corresponding $p_1,\ F_1,\ m_1,\ p_2,\ F_2,\ m_2$. If $m_1 < m_2$ (or $m_1 > m_2$), using the partial integration on $\exp(m_p(x))F_1(x)$ (or $\exp(m_{p2}(x))F_2(x)$) we obtain the representation of f_1 and f_2 with $m_1 = m_2$. If $p_1 < p_2$ we can put

$$\exp(m_{p1}(x))F_1(x) = \exp(m_{p2}(x))\tilde{F}_1(x) \text{ where } \tilde{F}_1(x) = \exp(m_{p1}(x) - m_{p2}(x))F_1(x).$$

If $p_1 > p_2$, we make the similar change on f_2. In any case we obtain that arbitrary two elements from H' have the representation of the form (9) with the same p and m. From that it follows the assertion of this theorem.

In the space H' we introduce the convergent structure in the following way:

$f_n \to f$ in H' iff there exists a sequence of locally integrable functions (F_n), a locally integrable function $F(x)$, $p \in N$ and $k \in N_0$ such that

$$D^k_{n_p} F_n = f_n, \quad D^k_{n_p} F = f, \tag{15}$$

and a sequence $F_n \exp(-n_p(x))$ is from L^2 and in L^2 norm converges to $F\exp(-n_p(x))$.

Theorem 3. *A sequence (f_n) from H' converges in H' to $f \in H'$ iff there exists a sequence of bounded continuous functions $(F_n(x))$, bounded continuous function $F(x)$, $p \in N$ and $m \in N_0$ such that*

$$f_n(x) = (F_n(x)\exp(m_p)x)))^{(m)}, \ \ f(x) = (F(x)\exp(m_p(x)))^{(m)} \tag{16}$$

and $F_n(x)$ converges to $F(x)$ for every $x \in R$.

Proof. If (15) holds, let us put $F_{1n}(x) = S_{n_p}F_n(x)$ and $F_1(x) = S_{n_p}F(x)$. It follows that

$$D^{k+1}_{n_p}(F_{1n} - F_1) = \sum_{l=0}^{k+1} c_l (N_l(x)F_{1n}(x) - F_1(x)))^{(l)}$$

where $N_l(x)$ are functions described in the proof of the preceding theorem.

From the inequality

$$| F_{1n}(x) - F_1(x) | \le \exp(-n_p(x))\Big(\int_0^x | F_n(t) - F(t) |^2 \exp(-2n_p(t))dt\Big)^{1/2} \cdot$$
$$\cdot\Big(\int_0^x \exp(4n_p(t))dt\Big)^{1/2}$$

it follows that $F_{1n}(x) \to F_1(x)$ for every $x \in R$. Using the same fact as in the first part of the proof of Theorem 1., we can show that f_n and f satisfy (16).

Let us show that (15) follows from (16).

For the suitable p_0 from (7) it follows that f_n and f are of the form

$$f_n(x) = F_n(\exp(-n_{p0}))^{(m)}, \ n \in N, \ \text{ and } \ f = (F\exp(-n_{p0}))^{(m)}$$

where (F_n) is a sequence of bounded continuous functions and f is a bounded continuous function. In the same way as in the second part of the proof of Theorem., we can show that $f_n(x)$, $n \in N$, and $f(x)$ are the finite sum of the expressions of the form

$$D^j_{n_p0}(S^r_{n_p0}(n^{(l)}_{p0}(x))^s \tilde{\tilde{F}}),\ n \in N; r,\ l,\ s \leq j \leq m;\ \text{ and}$$
$$D^j_{n_p0}(S^r_{n_p0}(n^{(l)}_{n_p0}(x))^s \tilde{\tilde{F}}).$$

The sequence $(\exp(-n_{p0})S^r_{n_p0}((n^{(l)}_{p0}(x)^s \tilde{\tilde{F}}))$ is from L^2 and $\exp(-n_{p0})S^r_{n_p0}(n^{(l)}_{p0}(x))^s$ $\tilde{\tilde{F}}) \in L^2$. Using Lebesgue Dominant Convergence Theorem and Lemma 1 (ii), we obtain that this sequence converges in L^2 to the element

$$\exp(-n_{p0}(x))S^r_{n_p0}((n^{(l)}_{n_p0}(x))^s \tilde{\tilde{F}}).$$

From the identities of the form (14) and Lemma 1 (ii) the assertion follows.

Remark 2. From Theorems 1. and 2. it follows that the space H' is identical to the space $H'\{\exp(m_p(x))\}$ (from [**6**]), which is the K'-type space introduced in [**2**]. Theorem 3. shows that the introduced convergent structure in H' is the same as the weak convergent structure in $H'\{\exp(m_p(x))\}$. In fact we have to verify that for the sequence $(\exp(m_p(x)))$, the condition (F) from [**4**] is satisfied and after that to use Theorem 7 (iv) from [**4**]

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