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ON EXTRAPOLATION OF MOVING AVERAGE AND AUTOREGRESSIVE PROCESSES

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1. Introduction

Let $X(s) = (X_1(s), \dots, X_n(s))$, $s \in R$, be a multidimensional wide sense stationary random process with the mean value zero, spectral density matrix $\|f^x_{jk}(\lambda)\|$ and spectral process $Z^x(\lambda) = (Z^x_1(\lambda), \dots, Z^x_n(\lambda))$, $\lambda \in R$. Suppose we know the values of the process $X(s)$ on the finite interval $[t-T, t]$. The problem of linear extrapolation of stationary random process $X(s)$ at the point $t+\tau$, $\tau > 0$, can be formulated as follows: Find the random variable

$$\tilde{X}_1(t,\tau,T) = \sum_{k=1}^{n} \int_{-\infty}^{+\infty} e^{it\lambda} \Phi_k(\lambda)\, dZ^X_k(\lambda) \tag{1.1}$$

which is the linear least-squares estimator of $X_1(t+\tau)$ given $X_k(s)$, $t-T \le s \le t$, $k = 1, 2, \dots, n$. The function $(\Phi_1(\lambda), \dots, \Phi_n(\lambda))$ will be called the spectral characteristic for extrapolation of the process $X(s)$ at the pont $t+\tau$. Let $H(X)$ denote the Hilbert space generated by $\{X_k(s), -\infty < s < +\infty,\ k = 1, 2, \dots, n\}$, and $H(X,t,T)$-the smallest Hilbert space spanned by $\{X_k(s)\ t-T \le s \le t$, $k = 1, 2 \dots, n\}$. Then, $\tilde{X}_1(t,\tau,T)$ is the projection of $X_1(t+\tau)$ into $H(X,t,T)$.

For the class of stationary random processes $X(s) = (X_1(s), \dots, X_n(s))$ having the nonsingular spectral density matrix $\|f_{jk}(\lambda)\|$, where all $f_{jk}(\lambda)$ are rational functions of λ, this extrapolation problem was studied in [**6**].

Now, let $X(s) = (X_1(s), \dots, X_n(s))$ and $Y(s) = (Y_1(s), \dots, Y_n(s))$ be two multidimensional stationary random processes satisfying the following equation

$$Y(s) = \sum_{\nu=0}^{N} a_\nu X(s - \nu\theta),\ a_0 = 1,\ a_\nu \in R,\ \theta > 0 \tag{1.2}$$

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and let the roots of the equation

$$\lambda^N + a_1\lambda^{N-1} + \cdots + a_{N-1}\lambda + a_N = 0 \tag{1.3}$$

be smaller than one in absolute value. Then,

$$X_k(s) = \sum_{\nu=0}^{\infty} c_\nu Y_k(s-\nu\theta),\ k = 1,2,\dots,n, \tag{1.4}$$

when the series on the right side of (1.4) converges in quadratic mean and the coefficients c_ν satisfy the homogeneous difference equations

$$a_0 c_k + a_1 c_{k-1} + \cdots + a_N c_{k-N} = 0,\ k \geq N \tag{1.5}$$

and the initial conditions

$$c_0 = 1,\ a_0 c_k + a_1 c_{k-1} + \cdots + a_k c_0 = 0,\ 0 < k < N. \tag{1.6}$$

If $\|f_{jk}^X(\lambda)\|$ and $\|f_{jk}^Y(\lambda)\|$ are the spectral matrices of the processes $X(s)$ and $Y(s)$, then we have

$$f_{jk}^Y(\lambda) = \left|\sum_{\nu=0}^{N} a_\nu e^{-i\nu\theta\lambda}\right|^2 f_{jk}^X(\lambda). \tag{1.7}$$

In this paper we shall find:

I) the linear least-squares estimator $\tilde{Y}_1(t,\tau,T)$ of $Y_1(t+\tau)$ given the values $Y_k(s)$, $t-T \leq s \leq t$, $k = 1,2,\dots,n$, if $X(s)$ is a nonsingular process with a rational spectrum,

II) the linear least-squares estimator $\tilde{X}_1(t,\tau,T)$ of $X_1(t+\tau)$ given the values $X_k(s)$, $t-T \leq s \leq t$, $k = 1,2,\dots,n$, if $Y(s)$ is a nonsingular stationary random process with a rational spectrum.

For the single processes $Y(s) = Y_1(s)$ and $X(s) = X_1(s)$ this problem was studied in [**3**].

The following lemma will be used:

Lemma 1. (Yaglom, A. M., [**6** 275–277]): *A function* $(\Phi_1(\lambda),\dots,\Phi_n(\lambda))$ *is the spectral characteristic for extrapolation of a stationary random process* $X(s)$ *at the point* $t+\tau$, $\tau > 0$, *given* $X_k(s)$, $t-T \leq s \leq t$, $k = 1,\dots,n$, *if and only if:*

$$1^\circ.\ \int_{-\infty}^{+\infty} |\,\Phi_k(\lambda)\,|^2\, f_{kk}^X(\lambda)\,d\lambda < \infty,\ \ k = 1,2,\dots,n \tag{1.8}$$

2°. *The functions*

$$\psi_k(\lambda) = \big(e^{i\tau\lambda} - \Phi_1(\lambda)\big) f_{1k}(\lambda) - \sum_{j=2}^{n} \Phi_j(\lambda) f_{jk}(\lambda),\ k = 1,2,\dots,n \tag{1.9}$$

can be represented in the form

$$\psi_k(\lambda) = \psi_k^{(1)}(\lambda) + e^{-i\lambda\tau}\psi_k^{(2)}(\lambda) \tag{1.10}$$

where: a_1) *the function* $\psi^1_{(k)}(\lambda)$ *is analytic in the upper half-plane and*

a_2) *as* $|\ \lambda\ | \to \infty$ *in the upper half-plane,* $\psi^1_{(k)}(\lambda)$ *falls off faster than* $|\ \lambda\ |^{-1-\varepsilon}$, $\varepsilon > 0$,

b_1) *the function* $\psi^{(2)}_k(\lambda)$ *is analytic in the lower half-plane and*

b_2) *as* $|\ \lambda\ | \to \infty$ *in the lower galf-plane,* $\psi^{(2)}_k(\lambda)$ *falls off faster than* $|\ \lambda\ |^{-1-\varepsilon}$, $\varepsilon > 0$.

3° *the functions* $\Phi_k(\lambda)$, $k = 1, 2, \ldots, n$, *are analytic functions represented in the form*

$$\Phi_k(\lambda) = \sum_\nu e^{i\tau_\nu \lambda} R_{k,\nu}(\lambda) \tag{1.11}$$

where $R_{k,\nu}(\lambda)$ *are rational functions and* $\tau_\nu \in [-T, 0]$.

For a stationary random process with the nonsingular spectral density matrix $\|f_{jk}(\lambda)\|$, where all $f_{jk}(\lambda)$ are rational functions of λ, we shall use the following notation

$$D(\lambda) = \det\|f_{jk}(\lambda)\| = \det\left\|\frac{Q_{jk}(\lambda)}{P_{jk}(\lambda)}\right\| = \frac{Q(\lambda)}{P(\lambda)},$$
$$P(\lambda) = (\lambda - \theta_1)\cdots(\lambda - \theta_L)(\lambda - \overline{\theta}_1)\cdots(\lambda - \overline{\theta}_L)$$

and we shall denote the degrees of polynomials $P_{jj}(\lambda)$, $Q_{jj}(\lambda)$, $P(\lambda)$, $Q(\lambda)$, by $2N_{jj}$, $2(N_{jj} - m_j)$, $2K$, $2L$, respectively.

2. Exstrapolation of moving average processes

THEOREM 2.1. *Let* $X(s) = (X_1(s), \ldots, X_n(s))$ *be a nonsingular stationary random process with a rational spectrum, and let* $Y(s)$ *be given by* (1.2) *where the roots of the equations* (1.3) *are less than one in absolute value. Suppose we know the values* $Y_k(s)$, $t - T \le s \le t$, $k = 1, 2, \ldots, n$, *and suppose* T/θ *is not an integer. Denote* $[T/\theta] = l$ *and* $[\tau/\theta] = S$.

If $S \ge N$, *then the spectral characteristic* $(\Phi^Y_1(\lambda), \ldots, \Phi^Y_n(\lambda))$ *for extrapolation of a stationary random process* $Y(s)$ *at the point* $t + \tau$ *has the following form:*

$$\Phi^Y_k(\lambda) = R^{(1)}_k(\lambda) \sum_{j=0}^{l} c^1_{kj} e^{-i\lambda j\theta} + e^{-i\lambda T} R^{(2)}_k(\lambda) \sum_{j-0}^{l} c^{(2)}_{kj} e^{i\lambda j\theta},\ k = 1, 2, \ldots, n \tag{2.1}$$

where

$$R^{(i)}_k(\lambda) = \frac{\omega^{(i)}_k(\lambda)}{(\lambda - \theta_1)\cdots(\lambda - \theta_L)(\lambda - \overline{\theta_1}) - \cdots(\lambda - \overline{\theta}_L)} \tag{2.2}$$

and $\omega^{(i)}_k(\lambda)$, $i = 1, 2$ *are the polynomials of the degree* $2L + m_k - 1$.

Proof. The functions $\Phi_k(\lambda)$, $k = 1, 2, \ldots, n$, given by (2.1) and (2.2) have the form (1.11) and satisfy the condition 1°. of Lemma 1. We shall define the coefficients $c_{kj}^{(1)}, c_{kj}^{(2)}$ and the coefficients of the polynomials $\omega_k^{(i)}(\lambda)$ so that these functions satisfy the conditions 2° and 3°.

Using (1.7), we have

$$\psi_m^Y(\lambda) = (e^{i\tau\lambda} - \Phi_1^Y(\lambda))f_{1m}^Y(\lambda) + \sum_{k=2}^{n} \Phi_k^Y(\lambda) f_{km}^Y(\lambda) =$$

$$= \left\{(e^{i\theta\lambda} - \Phi_1^Y(\lambda))f_{1m}^X(\lambda) + \sum_{k=2}^{n} \Phi_k^Y(\lambda) f_{km}^X(\lambda)\right\} \left| \sum_{j=0}^{N} a_j e^{-i\lambda j\theta} \right|^2. \tag{2.3}$$

Using the following equations

$$\left| \sum_{j=0}^{N} a_j e^{-i\lambda j\theta} \right|^2 = \sum_{j=-N}^{N} b_j e^{i\lambda j\theta},\ \lambda \in R \tag{2.4}$$

$$\sum_{j=0}^{l} c_{kj}^{(1)} e^{-i\lambda j\theta} \sum_{j=-N}^{N} b_j e^{i\lambda j\theta} = \sum_{j=-N-l}^{N} \alpha_{kj} e^{i\lambda j\theta} \tag{2.5}$$

$$\sum_{j=0}^{l} c_{kj}^{(2)} e^{i\lambda j\theta} \sum_{j=-N}^{N} b_j e^{i\lambda j\theta} = \sum_{j=-N}^{N+l} \beta_{kj} e^{i\lambda j\theta} \tag{2.6}$$

the functions $\psi_m^Y(\lambda),\ m = 1, 2, \dots, n$ can be represented in the form

$$\psi_m^Y(\lambda) = \sum_{j=-N}^{N} b_j e^{i\lambda(\tau+j\theta)} f_{1m}^X(\lambda) - \sum_{k=1}^{n} \psi_{k,1}(\lambda) f_{k,m}^X(\lambda) -$$

$$-e^{-i\lambda T} \sum_{k=1}^{n} \psi_{k,2}(\lambda) f_{km}^X(\lambda) - \sum_{k=1}^{n} \chi_k(\lambda) f_{km}^X(\lambda) \tag{2.7}$$

where

$$\psi_{k,1}(\lambda) = R_k^{(1)}(\lambda) \sum_{j=0}^{N} \alpha_{kj} e^{i\lambda j\theta} + R_k^{(2)}(\lambda) \sum_{j=l+1}^{l+N} \beta_{kj} e^{i\lambda(-T+j\theta)} \tag{2.8}$$

$$\psi_{k,2}(\lambda) = R_k^{(1)}(\lambda) \sum_{j=-N-l}^{-l-1} \alpha_{kj} e^{i\lambda(T+j\theta)} + R_k^{(2)}(\lambda) \sum_{j=-N}^{0} \beta_{kj} e^{i\lambda j\theta} \tag{2.9}$$

$$\chi_k(\lambda) = R_k^{(1)}(\lambda) \sum_{j=-l}^{-1} \alpha_{kj} e^{i\lambda j\theta} + e^{-i\lambda T} R_k^{(2)}(\lambda) \sum_{j=l}^{l} \beta_{kj} e^{i\lambda j\theta} \tag{2.10}$$

The condition $S \geq N$ implies the following inequalities

$$\tau + j\theta \geq 0,\ j = -N,\ -N+1, \dots, N.$$

If we define

$$\psi_m^{(1)}(\lambda) = \sum_{j=-N}^{N} b_j e^{i\lambda(\tau+j\theta)} f_{1m}^X(\lambda) - \sum_{k=1}^{n} \psi_{k,1}(\lambda) f_{km}^X(\lambda) \tag{2.11}$$

$$\psi_m^{(2)}(\lambda) = -\sum_{k=1}^{n} \psi_{k,2}(\lambda) f_{km}^X(\lambda) \tag{2.12}$$

and if we put

$$\chi_k(\lambda) = 0, \quad k = 1, 2, \ldots, n \tag{2.13}$$

then, the functions $\psi_m^Y(\lambda)$ will have the form (1.10), and the conditions a_2) and b_2) of Lemma 1. will be satisfied.

The equations (2.13) imply the following equations

$$\alpha_{k,j} = 0, \quad j = -l,\ -l+1, \ldots, -1, \quad k = 1, 2, \ldots, n \tag{2.14}$$

$$\beta_{kj} = 0, \quad j = 1, 2, \ldots, l, \quad k = 1, 2, \ldots, n \tag{2.15}$$

If we put $c_{k0}^{(1)} = c_{k0}^{(2)} = 1,\ k = 1, 2, \ldots, n$, then, from (2.14) and (2.15) we can determine

$$c_{1j}^{(1)} = c_{2j}^{(1)} = \cdots = c_{nj}^{(1)} (= c_j^{(1)}),\ j = 1, 2, \ldots, l, \tag{2.16}$$

$$c_{1j}^{(2)} = c_{2j}^{(2)} = \cdots = c_{nj}^{(2)} (= c_j^{(2)}),\ j = 1, 2, \ldots, l, \tag{2.17}$$

The coefficients of the polynomials $\omega_k^{(i)}(\lambda),\ i = 1, 2,\ k = 1, \ldots, n$ (there are $4nL + 2(m_1 + m_2 + \cdots + m_n)$ of such coefficients) can be found as in [**6**] so that the conditions $a_1), a_2)$ and 3° are satisfied.

Remark 1: If we consider the following equations

$$Y_j(s) = \sum_{\nu=0}^{N} a_\nu^{(j)} X(s - \nu\theta),\ \ a_0^{(j)} = 1,\ j = 1, 2, \ldots, n$$

instead of (1.2) and if the roots of the equations

$$\lambda^N + a_1^{(j)} \lambda^{N-1} + \cdots + a_{N-1}^{(j)} \lambda + a_N^{(j)} = 0,\ j = 1, 2, \ldots, n$$

are less than one in absolute value, then the spectral characteristic has the form (2.1). In this case the coefficients $c_{kj}^{(1)}, c_{kj}^{(2)},\ k = 1, 2, \ldots, n;\ j = 1, 2, \ldots, l$ may be obtained from the equations (2.14) and (2.15) but the equalities (2.16) and (2.17) do not hold.

THEOREM 2.2. *Let the assumptions be as in Theorem* 2.1. *If $S < N$, the spectral characteristic $(\Phi_1^Y(\lambda), \ldots, \Phi_n^Y(\lambda))$ has the form:*

$$\Phi_1^Y(\lambda) = R_1^{(1)}(\lambda) \sum_{j=0}^{l} c_{1j}^{(1)} e^{-i\lambda j\theta} + e^{-i\lambda T} R_1^{(2)}(\lambda) \sum_{j=0}^{l} c_{1j}^{(2)} e^{i\lambda j\theta} + \sum_{\nu\in A} c_\nu^{(3)} e^{i\lambda(\tau+\nu\theta)} \tag{2.18}$$

$$\Phi_k^Y(\lambda) = R_k^{(1)}(\lambda) \sum_{j=0}^{l} c_{kj}^{(1)} e^{-i\lambda j\theta} + e^{-i\lambda T} R_k^{(2)}(\lambda) \sum_{j=0}^{l} c_{kj}^{(2)} e^{i\lambda j\theta},\ k = 2, \ldots, n \tag{2.19}$$

where $A = \{\nu \mid -T < \tau + \nu\theta < 0\}$ *and the functions* $R_k^{(i)}(\lambda)$ *are as in Theorem* 2.1.

Proof. In this case we have

$$\psi_m^Y(\lambda) = \sum_{j=-N}^{N} b^j e^{i\lambda(\tau+j\theta)} f_{1m}^X(\lambda) - \sum_{\nu\in A} c_\nu^{(3)} e^{i\lambda(\tau+\nu\theta)} \sum_{j=-N}^{N} b^j e^{i\lambda j\theta} f_{1m}^X(\lambda)$$
$$- \sum_{k=1}^{n} \psi_{k,1}(\lambda) f_{km}^X(\lambda) - e^{-i\lambda T} \sum_{k=2}^{n} \psi_{k,2}(\lambda) f_{km}^X(\lambda) - \sum_{k=1}^{n} \chi_k(\lambda) f_{km}^X(\lambda) \quad (2.20)$$

where $\psi_{k,1}(\lambda)$, $\psi_{k,2}(\lambda)$, $\chi_k(\lambda)$, where given by (2.8)—(2.10). We determine the coefficients $c_\nu^{(3)}$, $\nu \in A$, from the condition that the functions $e^{i\lambda(\tau+j\theta)}$, $-T < \tau + j\theta < 0$, are not included in the sum

$$\sum_{j=-N}^{N} b_j e^{i\lambda(\tau+j\theta)} f_{1m}^X(\lambda) - \sum_{\nu\in A} c_\nu^{(3)} e^{i\lambda(\tau+\nu\theta)} \sum_{j=-N}^{N} b_j e^{i\lambda j\theta} f_{1m}^X(\lambda).$$

Then, let us represent this sum as $\sum_1 + \sum_2$ where the functions $e^{i\lambda(\tau+j\theta)}$, $\tau + j\theta \geq 0$, are included in $\sum_1$ and the functions $e^{i\lambda(\tau+j\theta)}$, $\tau + j\theta \leq -T$, are included in $\sum_2$. If we define

$$\psi_m^{(1)}(\lambda) = \sum_1 - \sum_{k=1}^{n} \psi_{k,1}(\lambda) f_{km}^X(\lambda) \quad (2.21)$$

$$\psi_m^{(2)}(\lambda) = e^{i\lambda T} \sum_2 - \sum_{k=1}^{n} \psi_{k,2}(\lambda) f_{km}^X(\lambda) \quad (2.22)$$

and if we put again $\chi_k(\lambda) = 0$, $k = 1, 2, \dots, n$, the functions $\psi_m(\lambda)$ will have the form (1.10), and the proof is completed as in the previous case.

Remark. The function $\Phi_1^Y(\lambda)$ given by (2.1), is also given by (2.18), where $A = \{\nu \mid -T < \tau + \nu\theta < 0\} = \varnothing$.

Corollary. *If* $T = l\theta$, *we have*

$$\Phi_1^Y(\lambda) = R_1(\lambda) \sum_{j=0}^{l} c_{1j} e^{-i\lambda j\theta} + \sum_{\nu\in A} k_\nu e^{i\lambda(\tau+\nu\theta)} \quad (2.23)$$

$$\Phi_k^Y(\lambda) = R_k(\lambda) \sum_{j=0}^{l} c_{kj} e^{-i\lambda j\theta} k = 2, \dots, n. \quad (2.24)$$

This form of the function $(\Phi_1^Y(\lambda), \dots, \Phi_n^Y(\lambda))$ is obtained if $T \to l\theta$ in (2.18) and (2.19).

Theorem 2.3. *Let the assumptions be as in Theorem* 2.1. *Then:*

a) If T/θ is not an integer, we have

$$\tilde{Y}_1(t,\tau,T)=\sum_{k=1}^{n}\Big\{\sum_{j=0}^{m_k-1}\sum_{j=0}^{l}[A_{kj}^{(\nu)}Y_k^{(\nu)}(t-j\theta)+B_{kj}^{(\nu)}Y_k^{(\nu)}(t-T+j\theta)]+$$
$$+\int_0^T w_k(s)Y_k(t-s)\,ds\Big\}+\sum_{\nu\in A}c_\nu^{(3)}Y_1(t+\tau+\nu\theta) \qquad (2.25)$$

b) If $T=l\theta$, $\tilde{Y}_1(t,\tau,T)$ will have the form

$$\tilde{Y}_1(t,\tau,T)=\sum_{k=1}^{n}\Big\{\sum_{\nu=0}^{m_k-1}\sum_{j=0}^{l}c_{kj}^{(\nu)}Y_k^{(\nu)}(t-j\theta)+$$
$$+\int_0^T w_k(s)Y_k(t-s)\,ds\Big\}+\sum_{\nu\in A}k_\nu Y_1(t+\tau+\nu\theta) \qquad (2.26)$$

Proof. a) After separating the polynomials from the rational functions $R_k^{(i)}(\lambda)$, $i=1,2,\ k=1,2,\dots,n$, the functions $\Phi_k^Y(\lambda)$, $k=1,\dots,n$, can be represented in the form

$$\Phi_1^Y(\lambda)=\sum_{\nu=0}^{m_k-1}\sum_{j=0}^{l}\{A_{1j}^{(\nu)}e^{i\lambda(-j\theta)}+B_{1j}^{(\nu)}e^{i\lambda(-T+j\theta)}\}(i\lambda)^\nu+\varphi_1(\lambda)+\sum_{\nu\in A}c_\nu^{(3)}e^{i\lambda(\tau+\nu\theta)} \qquad (2.27)$$

$$\Phi_k^Y(\lambda)=\sum_{\nu=0}^{m_k-1}\sum_{j=0}^{l}\{A_{kj}^{(\nu)}e^{i\lambda(-j\theta)}+B_{kj}^{(\nu)}e^{i\lambda(-T+j\theta)}\}(i\lambda)^\nu+\varphi_k(\lambda),\quad k=2,\dots,n. \qquad (2.28)$$

Then, $\int_{-\infty}^{+\infty}|\varphi_k(\lambda)|^2\,d\lambda<\infty$, $k=1,\dots,n$, and as $|\lambda|\to\infty$ in the lower halfplane, the functions $\varphi_k(\lambda)$ fall off not slower than $|\lambda|^{-1}$, and as $|\lambda|\to\infty$ in the upper half-plane, they behave as $|\lambda|^{-k}e^{TIm\lambda}$, $k\ge 1$. We can easily see that the Fourier transform $W_k(s)$ of $\varphi_k(\lambda)$ is equal to zero if $s\in(-\infty,-T]\cup[0,+\infty)$ and consequently we have

$$\varphi_k(\lambda)=\int_0^T e^{-i\lambda s}w_k(s)\,ds,\ k=1,2,\dots,n \qquad (2.29)$$

where $W_k(-s)=2\pi\cdot w(s)$. The formula (2.25) can be obtained from the equations (1.1), (2.27), (2.28) and (2.29).

3. Extrapolation of autoregressive processes

THEOREM 3.1. *Let $Y(s)=(Y_1(s),\dots,Y_n(s))$ be a nonsingular stationary random process with a rational spectrum, and $X(s)$ the process given by* (1.2), *where* *the roots of the equation* (1.3) *are less than one in absolute value. Suppose we know the values* $X_k(s) t-T \leq s \leq t$, $k=1,2,\dots,n$, *and* T/θ *is not an integer. Denote* $[T/\theta]=l$, $[\tau/\theta]=S$. *Then, the spectral characteristic* $(\Phi_1^X(\lambda),\dots,\Phi_n^X(\lambda))$ *for extrapolation of the stationary random process* $X(s)$ *at the point* $t+\tau$, $\tau>0$ *has the following form:*

a) If $0 \leq l < N$, *then*

$$\Phi_1^X(\lambda)=R_1^{(1)}(\lambda)\sum_{j=0}^{l} c_j^{(1)} e^{-i\lambda j\theta}+e^{-i\lambda T}R_1^{(2)}(\lambda)\sum_{j=0}^{l} c_j^{(2)} e^{i\lambda j\theta}+\sum_{\nu\in A} c_\nu^{(3)} e^{i\lambda(\tau-\nu\theta)} \tag{3.1}$$

$$\Phi_k^X(\lambda)=R_k^{(1)}(\lambda)\sum_{j=0}^{l} c_j^{(1)} e^{-i\lambda j\theta}+e^{-i\lambda T}R_1^{(2)}(\lambda)\sum_{j=0}^{l} c_j^{(2)} e^{i\lambda j\theta}, k=2,\dots,n. \tag{3.2}$$

b) If $l \geq N$, *then*

$$\Phi_1^X(\lambda)=R_1^{(1)}(\lambda)\sum_{\nu=0}^{N} a_\nu e^{-i\lambda\nu\theta}+e^{-i\lambda T}R_1^{(2)}(\lambda)\sum_{\nu=0}^{N} a_\nu e^{i\lambda\nu\theta}+\sum_{\nu\in B} c_\nu^{(3)} e^{i\lambda(\tau-\nu\theta)} \tag{3.3}$$

$$\Phi_k^X(\lambda)=R_k^{(1)}(\lambda)\sum_{\nu=0}^{N} a_\nu e^{-i\lambda\nu\theta}+e^{-i\lambda T}R_k^{(2)}(\lambda)\sum_{\nu=0}^{N} a_\nu e^{i\lambda\nu\theta}, k=2,\dots,n. \tag{3.4}$$

The functions $R_k^{(i)}(\lambda)$ *are rational functions as in Theorem* 2.1 *and*

$$A=\{\nu \mid -T<\tau-\nu\theta<0\},\quad B\{\nu \mid -T<\tau-\nu\theta,\ \nu\leq N\}.$$

Theorem 3.1 can be proved in the same way as Theorems 2.1 and 2.2 and then we have the following result:

THEOREM 3.2. *Under the assumptions of the Theorem* 3.1. *we have:*

a) If $0 \leq l < N$, *then the linear least-squares estimator* $\tilde{X}_1(t,\tau,T)$ *of* X_1 $(t+\tau)$ *has the following form*

$$\tilde{X}_1(t,\tau,T)=\sum_{k=1}^{n}\Big\{\sum_{\nu=0}^{m_k-1}\sum_{j=0}^{l}[A_{kj}^{(\nu)}X_k^{(\nu)}(t-j\theta)+B_{kj}^{(\nu)}X_k^{(\nu)}(t-T+j\theta)]+ + \int_o^T w_k(s)X_k(t-s)\,ds\Big\}+\sum_{\nu\in A} c_\nu^{(3)} X_1(t+\tau-\nu\theta). \tag{3.5}$$

b) If $l \geq N$, *then*

$$\tilde{X}_1(t,\tau,T)=\sum_{k=1}^{n}\Big\{\sum_{\nu=0}^{m_k-1}\sum_{j=0}^{l}[A_{kj}^{(\nu)}X_k^{(\nu)}(t-j\theta)+B_{kj}^{(\nu)}X_k^{(\nu)}(t-T+j\theta)]+ + \int_o^T w_k(s)X_k(t-s)\,ds\Big\}+\sum_{\nu\in B} c_\nu^{(3)} X_1(t+\tau-\nu\theta). \tag{3.6}$$

Remark 2: If $m_1 = m_2 = \cdots = m = 1$ then the predictor formulae (2.25), (2.26), (3.5) and (3.6) would not involve differentiation.

Example. Let $x_1(t)$ and $x_2(t)$ be two independent stationary random processes with the rational spectral densities $f_1(\lambda) = (\lambda^2+1)^{-1},\ f_2(\lambda) = (\lambda^2+4)^{-1}$ and define $X_1(t) = x_1(t) + x_2(t),\ X_2(t) = x_1(t) + x_2(t)$. Then, $(X_1(t), X_2(t))$ is a stationary random process with the spectral density matrix

$$\begin{bmatrix} (\lambda^2+1)^{-1} + (\lambda^2+4)^{-1} & (\lambda^2+1)^{-1} - (\lambda^2+4)^{-1} \\ (\lambda^2+1)^{-1} - (\lambda^2+4)^{-1} & (\lambda^2+1)^{-1} + (\lambda^2+4)^{-1} \end{bmatrix}$$

and $D(\lambda) = 4(\lambda^2+1)^{-1}(\lambda^2+4)^{-1},\ 2L = 0,\ 2k = 4,\ m_1 = m_2 = 1$.

Now, let $Y(t)$ be the process given by $Y(t) = X(t) - \beta X(t-1),\ \mid \beta \mid < 1$, and suppose we know the values of the process $Y(t)$ in the interval $[-1.5, 0]$. If we apply Theorem 2.1. we can find for $\tau > 1$:

$$\begin{aligned} &N = 1,\ n = 2,\ l = 1,\ S > 1,\ 2L + m_k - 1 = 0 \\ &R_1^{(1)}(\lambda) = K_1,\ R_1^{(2)}(\lambda) = K_2,\ R_2^{(1)}(\lambda) = k_1,\ R_2^{(2)}(\lambda) = k_2 \\ &\Phi_1^Y(\lambda) = K_1(1 + c_{11}^{(1)} e^{-i\lambda}) + e^{-3i\lambda/2} K_2 (1 + c_{11}^{(2)} e^{i\lambda}) \\ &\Phi_2^Y(\lambda) = k_1(1 + c_{21}^{(1)} e^{-i\lambda}) + e^{-3i\lambda/2} k_2 (1 + c_{21}^{(2)} e^{i\lambda}) \end{aligned}$$

The equalities (2.5) and (2.6) become

$$\begin{aligned} &(1 + c_{k_1}^{(1)} e^{-i\lambda})(-\beta e^{-i\lambda} + (1+\beta^2) - \beta e^{i\lambda}) = -\beta c_{k_1}^{(1)} e^{-2i\lambda} + \\ &\quad + (-\beta + (1+\beta^2) c_{k_1}^{(1)}) e^{-i\lambda} + (1 + \beta^2 - \beta c_{k_1}^{(1)}) - \beta e^{-i\lambda} \\ &(1 + c_{k_1}^{(2)} e^{i\lambda})(-\beta e^{-i\lambda} + 1 + \beta^2 - \beta e^{i\lambda}) = -\beta e^{-i\lambda} + \\ &\quad + (1 + \beta^2 - \beta c_{k_1}^{(2)}) + (-\beta + (1+\beta^2) c_{k_1}^{(2)}) e^{i\lambda} - \beta c_{k_1}^{(2)} e^{2i\lambda}. \end{aligned}$$

From the equations (2.14) and (2.15) we can find

$$c_{11}^{(1)} = c_{21}^{(1)} - \frac{\beta^2}{1+\beta^2},\ \ c_{11}^{(2)} = c_{21}^{(2)} = \frac{\beta}{1+\beta^2}$$

and consequently

$$\begin{aligned} \Phi_1^Y(\lambda) &= K_1\left(1 + \frac{\beta}{1+\beta^2} e^{-i\lambda}\right) + e^{-3i\lambda/2} K_2 \left(1 + \frac{\beta}{1+\beta^2} e^{i\lambda}\right) \\ \Phi_2^Y(\lambda) &= k_1\left(1 + \frac{\beta}{1+\beta^2} e^{-i\lambda}\right) + e^{-3i\lambda/2} k_2 \left(1 + \frac{\beta}{1+\beta^2} e^{i\lambda}\right) \end{aligned}$$

$$\psi_1^{(1)}(\lambda) =\{-\beta e^{i\lambda(\tau-1)} + (1+\beta^2)e^{i\lambda\tau} - \beta e^{i\lambda(\tau+1)}\}\{(\lambda^2+1)^{-2} + (\lambda^2+4)^{-1}\}$$
$$-\left\{K_1\left(1+\beta^2 - \frac{\beta^2}{1+\beta^2} - \beta e^{i\lambda}\right) - K_2\frac{\beta^2}{1+\beta^2}e^{i\lambda/2}\right\}\{(\lambda^2+1)^{-1} + (\lambda^2+4)^{-1}\}$$
$$-\left\{k_1\left(1+\beta^2 - \frac{\beta^2}{1+\beta^2} - \beta e^{i\lambda}\right) - k_2\frac{\beta^2}{1+\beta^2}e^{i\lambda/2}\right\}\{(\lambda^2+1)^{-1} - (\lambda^2+4)^{-1}\}$$
$$\psi_2^{(1)}(\lambda) =\{-\beta e^{i\lambda(\tau-1)} + (1+\beta^2)e^{i\lambda\tau} - \beta e^{i\lambda(\tau+1)}\}\{(\lambda^2+1)^{-1} - (\lambda^2+4)^{-1}\}$$
$$-\left\{K_1\left(1+\beta^2 - \frac{\beta^2}{1+\beta^2} - \beta e^{i\lambda}\right) - K_2\frac{\beta^2}{1+\beta^2}e^{i\lambda/2}\right\}\{(\lambda^2+1)^{-1} - (\lambda^2+4)^{-1}\}$$
$$-\left\{k_1\left(1+\beta^2 - \frac{\beta^2}{1+\beta^2} - \beta e^{i\lambda}\right) - k_2\frac{\beta^2}{1+\beta^2}e^{i\lambda/2}\right\}\{(\lambda^2+1)^{-1} + (\lambda^2+4)^{-1}\}$$
$$\psi_1^{(2)}(\lambda)=\left\{-K_1\frac{\beta^2}{1+\beta^2}e^{-i\lambda/2}+K_2\left(1+\beta^2 - \frac{\beta^2}{1+\beta^2} - \beta e^{-i\lambda}\right)\right\}\{(\lambda^2+1)^{-1}+(\lambda^2+4)^{-1}\}$$
$$+\left\{-k_1\frac{\beta^2}{1+\beta^2}e^{-i\lambda/2}+k_2\left(1+\beta^2 - \frac{\beta^2}{1+\beta^2} - \beta e^{-i\lambda}\right)\right\}\{(\lambda^2+1)^{-1}-(\lambda^2+4)^{-1}\}$$
$$\psi_2^{(2)}(\lambda)=\left\{-K_1\frac{\beta^2}{1+\beta^2}e^{-i\lambda/2}+K_2\left(1+\beta^2 - \frac{\beta^2}{1+\beta^2} - \beta e^{-i\lambda}\right)\right\}\{(\lambda^2+1)^{-1}-(\lambda^2+4)^{-1}\}$$
$$+\left\{-k_1\frac{\beta^2}{1+\beta^2}e^{-i\lambda/2}+k_2\left(1+\beta^2 - \frac{\beta^2}{1+\beta^2} - \beta e^{-i\lambda}\right)\right\}\{(\lambda^2+1)^{-1}+(\lambda^2+4)^{-1}\}$$

If we define $B_k^{(j)}(\lambda) = (\lambda^2+1)(\lambda^2+4)\psi_k^{(j)}$, $k;j = 1,2$, then the constants K_1, K_2, k_1, k_2 can be found from the equations

$$B_1^{(1)}(i) \equiv B_2^{(1)}(i) = 0, \quad B_1^{(1)}(2i) \equiv -B_2^{(1)}(2i) = 0$$
$$B_1^{(2)}(-i) \equiv B_2^{(2)}(-i) = 0, \quad B_1^{(2)}(-2i) \equiv -B_2^{(21)}(-2i) = 0$$

Then,

$$\tilde{Y}_1\left(0, \frac{3}{2}, \tau\right) = K_1Y_1(0) + \frac{\beta}{1+\beta^2}K_1Y_1(-1) + K^2Y_1\left(-\frac{3}{2}\right) + \frac{\beta}{1+\beta^2}K_2Y_1\left(-\frac{1}{2}\right)$$
$$+ k_1Y_2(0) + \frac{\beta}{1+\beta^2}k_1Y_2(-1) + k_2Y_2\left(-\frac{3}{2}\right) + \frac{\beta}{1+\beta^2}k_2Y_2\left(-\frac{1}{2}\right).$$

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