

DERIVATIONAL FORMULAS OF A SUBSPACE OF A GENERALIZED RIEMANNIAN SPACE

Svetislav M. Minčić

Summary. L. P. Einsenhart has defined in [1, 2] a generalized Riemannian space with nonsymmetric basical tensor, R. S. Maishra and M. Prvanović in [3, 4] have studied derivational formulas and Gauss-Codazzi equations in a subspace of a generalized Riemannian space. In [5, 6] we used 4 kinds of covariant derivation of a tensor in a subspaces of this space.

In the present work we obtain 4 kinds of derivational formulas, using the above mentioned kinds of derivation.

0. Introduction. Let V_N be a generalized Riemannian space with coordinates $y^\alpha (\alpha = 1, \dots, N)$ and a basical tensor $a_{\alpha\beta} (a_{\alpha\beta} \neq a_{\beta\alpha})$

The equations

$$(1) \quad y^\alpha = y^\alpha(x^1, \dots, x^M)$$

define a subspace V_M of the space V_N . The coordinates in the V_M are $x^i (i = 1, \dots, M)$, and the basical tensor is $g_{ij} (g_{ij} \neq g_{ji})$.

The next relations (see [1–3]) are valid;

$$(2) \quad a_{\alpha\beta} y_{,i}^\alpha y_{,j}^\beta = a_{\alpha\beta} t_i^\alpha t_j^\beta = g_{ij},$$

where the comma (,) signifies the usual partial derivation, and we have

$$(3) \quad \partial y^\alpha / \partial x^i = y_{,i}^\alpha = t_i^\alpha.$$

If we denote by $\underline{\alpha\beta}$ and $\overset{\vee}{\alpha\beta}$ the symmetrisation and antisymmetrisation over indices α, β and analogically in other cases, we have

$$(4a, b) \quad \underline{g_{ij}} = \underline{a_{\alpha\beta}} t_i^\alpha t_j^\beta, \quad \overset{\vee}{g_{ij}} = \overset{\vee}{a_{\alpha\beta}} t_i^\alpha t_j^\beta,$$

$$(5a, b) \quad \underline{g_{ij}} g^{jk} = \delta_j^k, \quad \underline{a_{\alpha\beta}} a^{\beta\gamma} = \delta_\alpha^\gamma.$$

The Christoffel symbols for V_N are

$$(6) \quad \Gamma_{\alpha,\beta\gamma} = (a_{\beta\alpha,\gamma} - a_{\beta\gamma,\alpha} + a_{\alpha\gamma,\beta})/2$$

$$(7) \quad \Gamma_{\beta\gamma}^\alpha = a^{\pi\alpha} \Gamma_{\pi,\beta\gamma} \quad (\Gamma_{\beta\gamma}^\alpha \neq \Gamma_{\gamma\beta}^\alpha)$$

and analogously for V_M . Further, for example, it is

$$(7') \quad a_{\underline{\alpha}\sigma}\Gamma_{\beta\gamma}^\sigma = a_{\underline{\alpha}\sigma}a^{\pi\sigma}\Gamma_{\pi,\beta,\gamma} = \delta_\alpha^\pi\Gamma_{\pi,\beta,\gamma} = \Gamma_{\alpha,\beta,\gamma}.$$

For unit, mutually orthogonal vectors $N_{(\varrho)}^\alpha$, which are also orthogonal on the V_M , we have

$$(8a) \quad a_{\underline{\alpha}\beta}N_{(\sigma)}^\beta = e_{(\varrho)}\delta_{\varrho\sigma} \quad (e_{(\varrho)} = \pm 1),$$

$$(8b) \quad a_{\underline{\alpha}\beta}N_{(\sigma)}^\alpha t_j^\beta = 0$$

where the Greek indices in the brackets take values from $M + 1$ to N and have not a tensor nature.

Because of non-symmetry of the connexion coefficients, we can define 4 kinds of covariant derivation in the generalized Riemannian space (see [5, 6]). For example, for a tensor $a_{\beta j}^{\alpha i}$, whose Greek indices are related to the space V_N , and Latin to the subspace V_M , for a covariant derivation on x^m , we have

(9a,b)

$$a_{\beta j}^{\alpha i}|_m^{\frac{1}{2}} = a_{\beta,j,m}^{\alpha i} + (\Gamma_{\pi\mu}^\alpha a_{\beta j}^{\pi i} - \Gamma_{\beta\mu}^\pi a_{\pi j}^{\alpha i})t_m^\mu + \Gamma_{pm}^i a_{\beta j}^{\alpha p} - \Gamma_{jm}^p a_{\beta p}^{\alpha i},$$

(9c, d)

$$a_{\beta j}^{\alpha i}|_m^{\frac{3}{4}} = a_{\beta,j,m}^{\alpha i} + (\Gamma_{\pi\mu}^\alpha a_{\beta j}^{\pi i} - \Gamma_{\beta\mu}^\pi a_{\pi j}^{\alpha i})t_m^\mu + \Gamma_{pm}^i a_{\beta j}^{\alpha p} - \Gamma_{jm}^p a_{\beta p}^{\alpha i},$$

If all indices of a tensor are related to the space, we can find its derivation by y^μ or by x^m . For example

$$(10a,b) \quad a_{\gamma|\mu}^{\alpha\beta} = a_{\gamma,\mu}^{\alpha\beta} + \Gamma_{\pi\mu}^\alpha a_{\gamma}^{\pi\beta} + \Gamma_{\pi\mu}^\beta a_{\gamma}^{\alpha\pi} - \Gamma_{\gamma\mu}^\pi a_{\pi}^{\alpha\beta},$$

$$(10c,d) \quad a_{\gamma|\mu}^{\alpha\beta} = a_{\gamma,\mu}^{\alpha\beta} + \Gamma_{\pi\mu}^\alpha a_{\gamma}^{\pi\beta} + \Gamma_{\pi\mu}^\beta a_{\gamma}^{\alpha\pi} - \Gamma_{\gamma\mu}^\pi a_{\pi}^{\alpha\beta},$$

$$(11) \quad a_{\gamma|_m}^{\alpha\beta} = a_{\gamma|\mu}^{\alpha\beta} t_m^\mu.$$

Further, using (6) we have

$$(12a,b) \quad \Gamma_{\alpha,\beta\gamma} + \Gamma_{\beta,\alpha\gamma} = a_{\underline{\alpha}\beta,\gamma}, \quad \Gamma_{\alpha,\beta\gamma} + \Gamma_{\gamma,\beta\alpha} = a_{\underline{\alpha}\gamma,\beta}$$

$$(12c) \quad \Gamma_{\alpha,\beta\gamma} + \Gamma_{\alpha,\gamma\beta} = a_{\underline{\alpha}\beta,\gamma} - a_{\underline{\beta}\gamma,\alpha} + a_{\underline{\gamma}\alpha,\beta}.$$

From (10a) and (7')

$$a_{\underline{\alpha}\beta|_1\gamma} = a_{\underline{\alpha}\beta,\gamma} - \Gamma_{\alpha\gamma}^\pi a_{\underline{\pi}\beta} - \Gamma_{\beta\gamma}^\pi a_{\underline{\alpha}\pi} = a_{\underline{\alpha}\beta,\gamma} - \Gamma_{\beta,\alpha\gamma} - \Gamma_{\alpha,\beta\gamma},$$

whence, using (12a) we obtain

$$(13) \quad a_{\underline{\alpha}\beta|_1\gamma} = 0.$$

It the same way, we prove that;

$$(13') \quad \begin{aligned} g_{\underline{ij}}|_m &= 0, \\ \underline{a_{\alpha\beta}}|_\gamma &= \underline{a_{\alpha\beta}}|_\gamma = 0, \quad \underline{a_{\alpha\beta}}|_4 = \underline{a_{\alpha\beta}}|_\gamma = 0, \\ g_{\underline{ij}}|_m &= g_{\underline{ij}}|_m = 0, \quad g_{\underline{ij}}|_m = g_{\underline{ij}}|_m = 0. \end{aligned}$$

Therefore, the following is valid:

$$(14a,b) \quad \underline{a_{\alpha\beta}}|_\gamma = 0, \quad g_{\underline{ij}}|_m = 0, \quad (\theta = 1, \dots, 4)$$

and, based on (11), it is

$$(14c) \quad \underline{a_{\alpha\beta}}|_\gamma = 0 \quad (\theta = 1, \dots, 4).$$

1. First derivational formulas. By derivational formulas one expresses covariant derivatives of the tensors t_i^α , $N_{(\theta)}^\alpha$ as linear combinations of these tensors.

From (9a-d) we have

$$(15a) \quad t_{\underline{i}}^\alpha|_m = t_{i,m}^\alpha + \Gamma_{\pi\mu}^\alpha t_m^\mu t_i^\pi - \Gamma_{im}^p t_p^\alpha,$$

$$(15b) \quad t_{\underline{i}}^\alpha|_m = t_{i,m}^\alpha + \Gamma_{\mu\pi}^\alpha t_m^\mu t_i^\pi - \Gamma_{mi}^p t_p^\alpha,$$

$$(15c) \quad t_{\underline{i}}^\alpha|_m = t_{i,m}^\alpha + \Gamma_{\pi\mu}^\alpha t_m^\mu t_i^\pi - \Gamma_{mi}^p t_p^\alpha,$$

$$(15d) \quad t_{\underline{i}}^\alpha|_m = t_{i,m}^\alpha + \Gamma_{\mu\pi}^\alpha t_m^\mu t_i^\pi - \Gamma_{im}^p t_p^\alpha,$$

and we can interpret the tensors $t_{\underline{i}}^\alpha|_m$ ($\theta = 1, \dots, 4$) in the following manner:

$$(16) \quad t_{\underline{i}}^\alpha|_m = \Phi_{im}^p t_p^\alpha + \sum_{\varrho=M+1}^N \Omega_{(\varrho)im} N_{(\varrho)}^\alpha$$

For $\theta = 1, \dots, 4$ we obtain from (16) 4 kinds of **I** derivational formula of the subspace V_M of the generalized Riemannian space V_N .

Multiplying the last equation by $a_{\alpha\beta} t_h^\beta$ and using (4a) and (8b) be obtain

$$(17) \quad \underline{a_{\alpha\beta}} t_{\underline{i}}^\alpha|_m t_h^\beta = \Phi_{im}^p \underline{a_{\alpha\beta}} t_p^\alpha t_h^\beta = \Phi_{im}^p g_{ph} = \Phi_{him}, \quad \text{i.e.}$$

$$(17') \quad \Phi_{im}^h = \Phi_{qim}^h g^{hq} = \underline{a_{\alpha\beta}} t_{\underline{i}}^\alpha|_m t_q^\beta g^{hq}.$$

Multiplying the equation (16) by $a_{\alpha\beta} N_{(\sigma)}^\beta$ based on (8a, b) it follows

$$\underline{a_{\alpha\beta}} t_{\underline{i}}^\alpha|_m N_{(\sigma)}^\beta = e_{(\sigma)} \Omega_{(\sigma)im} \quad (e_{(\sigma)} = \pm 1),$$

whence

$$(18) \quad \Omega_{\theta(\sigma)im} = e_{(\sigma)} \underline{a_{\alpha\beta}} t_{i|m}^{\alpha} N_{(\sigma)}^{\beta}.$$

We shall prove now that the tensors $\Phi_{\theta im}^h$ are antisymmetric on lower indices, and $\Phi_{\theta him}$ on all indices. Since

$$(19) \quad t_{m,l}^{\alpha} = t_{i,m}^{\alpha} = y_{,im}^{\alpha} = \partial^2 y^{\alpha} \setminus \partial x^i \partial x^m,$$

we conclude from (15a-d) that

$$(20) \quad \underline{t_{i|m}^{\alpha}} = (t_{i,|m}^{\alpha} + t_{m|i}^{\alpha})/2 = t_{i,m}^{\alpha} + \Gamma_{\pi\mu}^{\alpha} t_m^{\mu} t_i^{\pi} - \Gamma_{im}^p t_p^{\alpha}.$$

If, on the same manifold on which generalized Riemannian space V_N and its subspace V_M are defined, we define a usual Riemannian space \overline{V}_N and its subspace \overline{V}_M using as basal tensors $\underline{a_{\alpha\beta}}$ and $\underline{g_{ij}}$ the equation (2) becomes

$$(20') \quad t_{i|m}^{\alpha} = t_{i;m}^{\alpha}$$

where by the semicolon (;) we denote covariant derivation by using symmetric connexion coefficients $\Gamma_{\beta\gamma}^{\alpha}$ respectively Γ_{jk}^i obtained by $\underline{a_{\alpha\beta}}(g_{ij})$.

The derivational formulas of the subspace V_M (see [7, § 47]) are

$$(21a) \quad t_{i;m}^{\alpha} = \sum_{\varrho} \Omega_{(\varrho)im} N_{(\varrho)}^{\alpha}$$

$$(21b) \quad N_{(\sigma);m}^{\alpha} = -e_{(\sigma)} g^{ps} \Omega_{(\sigma)sm} t_p^{\alpha} + \sum_{\varrho} \Psi_{(\varrho\sigma)m} N_{(\varrho)}^{\alpha} \quad (\Psi_{\varrho\varrho m} = 0).$$

Using (18), (20)'

$$\Omega_{\theta(\sigma)\underline{im}} = e_{(\sigma)} \underline{a_{\alpha\beta}} \underline{t_{i|m}^{\alpha}} N_{(\sigma)}^{\beta} = e_{(\sigma)} \underline{a_{\alpha\beta}} t_{i;m}^{\alpha} N_{(\sigma)}^{\beta},$$

and from (21a):

$$(22) \quad \Omega_{\theta(\sigma)\underline{im}} = \Omega_{(\sigma)im}.$$

According to (21a), it is

$$(23) \quad \underline{a_{\alpha\beta}} t_{i;m}^{\alpha} t_h^{\beta} = \underline{a_{\alpha\beta}} \underline{t_{i|m}^{\alpha}} t_h^{\beta} = 0$$

and, based on (17'), we have

$$\Phi_{\theta im}^h = \underline{a_{\alpha\beta}} g^{hq} \underline{t_{i|m}^{\alpha}} t_q^{\beta}.$$

From this, according to (23), one obtains

$$(24) \quad \Phi_{\theta im}^h = 0 \text{ i.e.}$$

$$(24') \quad \Phi_{\theta im}^h = -\Phi_{\theta im}^h = \Phi_{\theta \vee im}^h,$$

which means that the tensors $\Phi_{\theta im}^h$ are antisymmetric on indices.

Taking into consideration (14 b, c) from (4a) there follows

$$(25a) \quad a_{\underline{\alpha\beta}} t_i^\alpha |_{\theta m} t_j^\beta + a_{\underline{\alpha\beta}} t_i^\alpha t_j^\beta |_{\theta m} = 0$$

By cyclical interchange of the indices we obtain

$$(25b) \quad a_{\underline{\alpha\beta}} t_j^\alpha |_{\theta m} t_i^\beta + a_{\underline{\alpha\beta}} t_j^\alpha t_m^\beta |_{\theta i} = 0,$$

$$(25c) \quad a_{\underline{\alpha\beta}} t_m^\alpha |_{\theta j} t_i^\beta + a_{\underline{\alpha\beta}} t_m^\alpha t_i^\beta |_{\theta j} = 0.$$

Adding (25b, c) and subtracting (25a), we get

$$a_{\underline{\alpha\beta}} (t_m^\alpha t_i^\beta |_{\theta j} + t_j^\alpha t_m^\beta |_{\theta i} + t_i^\alpha t_m^\beta |_{\theta j}) = 0.$$

From this and (23)

$$(26) \quad a_{\underline{\alpha\beta}} t_i^\alpha t_m^\beta |_{\theta j} + a_{\underline{\alpha\beta}} t_j^\alpha t_m^\beta |_{\theta i} = 0.$$

On the base of (16) and (24') we have

$$T_i^\alpha |_{\theta m} = \Phi_{\theta im}^p t_p^\alpha + \sum_{\varrho} \Omega_{\theta(\varrho)im} N_{(\varrho)}^\alpha = \Phi_{\theta im}^p t_p^\alpha + \sum_{\varrho} \Omega_{\theta(\varrho)im} N_{(\varrho)}^\alpha,$$

which put into (26), gives

$$a_{\underline{\alpha\beta}} t_i^\alpha \Phi_{\theta mj}^p t_p^\beta + a_{\underline{\alpha\beta}} t_j^\alpha \Phi_{\theta mi}^p t_p^\beta = 0, \text{ i.e.}$$

$$\underline{g}_{ip} \Phi_{\theta mj}^p + \underline{g}_{jp} \Phi_{\theta mi}^p = 0 \Rightarrow \Phi_{\theta imj} = -\Phi_{\theta jmi}.$$

From here and (24'), we have

$$(27) \quad \Phi_{\theta imj} = -\Phi_{\theta jmi} = -\Phi_{\theta ijm} = -\Phi_{\theta mij}.$$

We see that tensors $\Phi_{\theta ijm}$ are antisymmetric on all pairs of indices.

2. Second derivational formulas. From (9a-d), for covariant derivation of normals we have

$$(28a) \quad N_{(\sigma)_1}^\alpha |_{\theta m} = N_{(\sigma)_3}^\alpha |_{\theta m} = N_{(\sigma)m}^\alpha + \Gamma_{\pi\mu}^\alpha t_m^\mu n_{(\sigma)}^\pi,$$

$$(28b) \quad N_{(\sigma)_2}^\alpha |_{\theta m} = N_{(\sigma)_4}^\alpha |_{\theta m} = N_{(\sigma)m}^\alpha + \Gamma_{\mu\pi}^\alpha t_m^\mu N_{(\sigma)}^\pi.$$

From (8a) we get

$$a_{\underline{\alpha}\underline{\beta}}N_{(\varrho)}^{\alpha}N_{(\varrho)}^{\beta} = e_{(\varrho)} \quad (e_{(\varrho)} = \pm 1),$$

and, based on (14c):

$$(29) \quad a_{\underline{\alpha}\underline{\beta}}N_{(\varrho)}^{\alpha}|_m N_{(\varrho)}^{\beta} + a_{\underline{\alpha}\underline{\beta}}N_{(\varrho)}^{\alpha}N_{(\varrho)}^{\beta}|_m = 0 \Rightarrow a_{\underline{\alpha}\underline{\beta}}N_{(\varrho)}^{\alpha}|_m N_{(\varrho)}^{\beta} = 0, \text{ i.e.} \\ N_{(\varrho)}^{\alpha}|_m \perp N_{(\varrho)}^{\alpha}.$$

We can express the tensors $N_{(\varrho)}^{\alpha}|_m$ in the following way:

$$(30) \quad N^{\alpha}(\sigma | m = B_{\theta}^p(\sigma)m + \sum_{\varrho} \Psi_{\varrho\sigma} N_{(\varrho)}^{\alpha}, \quad \Psi_{(\sigma\sigma)m} = 0,$$

where the condition $\Psi_{(\sigma\sigma)m} = 0$ follows from (29). From (28a, b) and (30)

$$(31a, b) \quad B_{1(\sigma)m}^h = B_{3(\sigma)m}^h, \quad B_{3(\sigma)m}^h = B_{4(\sigma)m}^h,$$

$$(32a, b) \quad \Psi_{1(\varrho\sigma)m} = \Psi_{3(\varrho\sigma)m}, \quad \Psi_{2(\varrho\sigma)m} = \Psi_{4(\varrho\sigma)m}.$$

In order to determine the tensors $B_{\theta(\sigma)m}^h$, we shall differentiate covariantly the relation (8b). Taking into account (14c), we obtain

$$a_{\underline{\alpha}\underline{\beta}}N_{(\sigma)}^{\alpha}|_m t_j^{\beta} + a_{\underline{\alpha}\underline{\beta}}N_{(\sigma)}^{\alpha}t_j^{\alpha}|_m = 0.$$

From here, by a change based on (30), (16) we obtain

$$(33) \quad a_{\underline{\alpha}\underline{\beta}}t_j^{\beta} \left(B_{\theta(\sigma)m}^p t_p^{\alpha} + \sum_{\varrho} \Psi_{\theta(\varrho\sigma)m} N_{(\varrho)}^{\alpha} \right) + a_{\underline{\alpha}\underline{\beta}}N_{(\sigma)}^{\alpha} \left(\Phi_{\theta jm}^p t_p^{\beta} + \sum_{\varrho} \Omega_{\theta(\varrho)jm} B_{(\varrho)}^{\beta} \right) = 0, \text{ i.e.} \\ g_{j\underline{p}} B_{\theta(\sigma)m}^p + e_{(\sigma)} \Omega_{(\sigma)jm} = 0.$$

Multiplying the previous relation by $g^{\underline{ij}}$, we get

$$\delta_p^i B_{\theta(\sigma)m}^p + e_{(\sigma)} g^{\underline{ij}} \Omega_{(\sigma)jm} = 0,$$

respectively

$$(34) \quad B_{\theta(\sigma)m}^i = -e_{(\sigma)} g^{\underline{is}} \Omega_{(\sigma)sm} = -e_{(\sigma)} \Omega_{\theta(\sigma)m}^i.$$

With respect to (31, 34) we conclude

$$(35a,b) \quad \Omega_{1(\sigma)m}^i = \Omega_{3(\sigma)m}^i, \quad \Omega_{2(\sigma)m}^i = \Omega_{4(\sigma)m}^i.$$

From (33)

$$(36) \quad \Omega_{(\sigma)im} = -e_{(\sigma)} g_{i\underline{p}} B_{(\sigma)m}^p,$$

and from this and (31a, b) we have

$$(35'a, b) \quad \Omega_{(\sigma)im} = \Omega_{3(\sigma)im}^i, \quad \Omega_{2(\sigma)im} = \Omega_{4(\sigma)im}^i.$$

Changing $B_{\theta(\sigma)m}^p$ based on (34) into (30), we obtain

$$(37) \quad N_{(\sigma)\theta}^{\alpha}|_m = -e_{(\sigma)}\Omega_{\theta(\sigma)m}^p t_p^{\alpha} + \sum_p \Psi_{\theta(\varrho\sigma)m} N_{(\varrho)}^{\alpha}, \quad \Psi_{(\sigma\sigma)m} = 0, \quad \text{i.e.}$$

$$(37') \quad N_{(\sigma)\theta}^{\alpha}|_m = -e_{(\sigma)}g_{\theta}^{ps}\Omega_{(\sigma)sm} t_p^{\alpha} + \sum_p \Psi_{\theta(\varrho\sigma)m} N_{(\varrho)}^{\alpha}, \quad \Psi_{(\sigma\sigma)m} = 0. \quad \text{i.e.}$$

From this and (28a, b) we obtain 2 kinds of the second derivational formula of the subspace V_M (for $\theta = 1, 2$).

We shall now investigate some properties of the tensors

$$\Psi_{\theta}^h{}_{im}, \quad \Psi_{\theta(\sigma)im}, \quad \Psi_{\theta(\varrho\sigma)m}.$$

From (37') and (8a, b) we have

$$a_{\alpha\beta} N_{(\sigma)\theta}^{\alpha}|_m N_{(\pi)}^{\beta} = \Psi_{\theta(\pi\sigma)m} a_{\alpha\beta} N_{(\pi)}^{\alpha} N_{(\sigma)}^{\beta} = e_{(\pi)} \Psi_{\theta(\pi\sigma)m},$$

from where it is

$$(38) \quad \Psi_{\theta(\pi\sigma)m} = e_{(\pi)} a_{\alpha\beta} N_{(\pi)}^{\alpha} N_{(\sigma)}^{\beta}|_m \quad (e_{(\pi)} = \pm 1).$$

On the other hand, by covariant differentiation of the relation

$$\begin{aligned} a_{\alpha\beta} N_{(\pi)}^{\alpha} N_{(\sigma)}^{\beta} &= e_{(\pi)} \delta_{\pi\sigma}, \quad \text{we get} \\ a_{\alpha\beta} \left(N_{(\pi)}^{\alpha}|_m N_{(\sigma)}^{\beta} + N_{(\pi)}^{\alpha} B_{(\sigma)\theta}^{\beta}|_m \right) &= 0, \end{aligned}$$

whence, from (28), we have

$$(39) \quad e_{(\pi)} \Psi_{\theta(\pi\sigma)m} + e_{(\sigma)} \Psi_{\theta(\sigma\pi)m} = 0, \quad \text{or}$$

$$(39') \quad \Psi_{\theta(\pi\sigma)m} = -e_{(\pi)} e_{(\sigma)} \Psi_{\theta(\sigma\pi)m} = \pm \Psi_{\theta(\sigma\pi)m}.$$

With regard to (8a), one obtains the sign $-$ in the case when both normals $N_{(\pi)}^{\alpha}$ and $N_{(\sigma)}^{\alpha}$ are either real or imaginary, and the sign $+$ in the case when one of these normals is real and the other is imaginary.

From the definitions (15a-d) it is

$$(40a, b) \quad t_{2i}^{\alpha}|_m = t_{m1}^{\alpha}|_i, \quad t_{4i}^{\alpha}|_m = t_{m3}^{\alpha}|_i,$$

and from this and (16), 24'), we have

$$(41a, b) \quad \Phi_{2im}^h = \Phi_{m1i}^h = -\Phi_{1im}^h, \quad \Phi_{4im}^h = \Phi_{m3i}^h = -\Phi_{3im}^h.$$

From (40a, b), (16) and (35'a, b)

$$(42) \quad \begin{aligned} \Omega_{2(\sigma)im} &= \Omega_{1(\sigma)mi} = \Omega_{3(\sigma)mi}, \quad \Omega_{4(\sigma)im} = \Omega_{3(\sigma)mi} = \Omega_{1(\sigma)mi}, \quad \text{i.e.} \\ \Omega_{1(\sigma)im} &= \Omega_{2(\sigma)mi} = \Omega_{3(\sigma)im} = \Omega_{4(\sigma)mi}. \end{aligned}$$

Further, using (15a-d)

$$(43a) \quad t_{i|m}^{\alpha} = t_{i;m}^{\alpha} + \Gamma_{\pi\sqrt{\nu}}^{\alpha} t_m^{\mu} t_i^{\pi} - \Gamma_{i\sqrt{\nu}}^p t_p^{\alpha},$$

$$(43b) \quad t_{i|m}^{\alpha} = t_{i;m}^{\alpha} + \Gamma_{\mu\sqrt{\pi}}^{\alpha} t_m^{\mu} t_i^{\pi} - \Gamma_{m\sqrt{\nu}}^p t_p^{\alpha},$$

$$(43c) \quad t_{i|m}^{\alpha} = t_{i;m}^{\alpha} + \Gamma_{\pi\sqrt{\mu}}^{\alpha} t_m^{\mu} t_i^{\pi} - \Gamma_{m\sqrt{\nu}}^p t_p^{\alpha},$$

$$(43d) \quad t_{i|m}^{\alpha} = t_{i;m}^{\alpha} + \Gamma_{\mu\sqrt{\pi}}^{\alpha} t_m^{\mu} t_i^{\pi} - \Gamma_{i\sqrt{\nu}}^p t_p^{\alpha},$$

From the previous equations we see that

$$(44) \quad \left(t_{i|m}^{\alpha} + t_{i|m}^{\alpha} \right) / 2 = \left(t_{i|m}^{\alpha} + t_{i|m}^{\alpha} \right) / 2 = t_{i;m}^{\alpha},$$

and, taking into account (16) and (21a), we obtain

$$\Phi_{1im}^h + \Phi_{2im}^h = \Phi_{3im}^h + \Phi_{4im}^h = 0,$$

which affirm the equations (41a, b). We have also

$$(45) \quad (\Omega_{1(\sigma)im} + \Omega_{2(\sigma)im}) / 2 = (\Omega_{3(\sigma)im} + \Omega_{4(\sigma)im}) / 2 = \Omega_{(\sigma)im}.$$

From (22), (45) one affirms (42).

If we make a change in (43a) using (16), (21a), (22), we get

$$(46) \quad \Phi_{1im}^p t_p^{\alpha} + \sum_p [\Omega_{1(\varrho)im} - \Omega_{1(\varrho)im}] N_{(\varrho)}^{\alpha} = \Gamma_{\pi\sqrt{\nu}}^{\alpha} t_i^{\pi} t_m^{\mu} - \Gamma_{i\sqrt{\nu}}^p t_p^{\alpha}.$$

Multiplying this equation by $a_{\alpha\beta} t_h^{\beta}$, we have

$$(47a) \quad \Phi_{1him} = a_{\alpha\beta} \Gamma_{\pi\sqrt{\nu}}^{\alpha} t_h^{\beta} t_i^{\pi} t_m^{\mu} - \Gamma_{h.i\sqrt{\nu}}.$$

In the same way, using (43b - d), we obtain

$$(47b) \quad \Phi_{2him} = a_{\alpha\beta} \Gamma_{\mu\sqrt{\pi}}^{\alpha} t_h^{\beta} t_i^{\pi} t_m^{\mu} - \Gamma_{h.m\sqrt{\nu}},$$

$$(47c) \quad \Phi_{3him} = a_{\alpha\beta} \Gamma_{\pi\sqrt{\mu}}^{\alpha} t_h^{\beta} t_i^{\pi} t_m^{\mu} - \Gamma_{h.m\sqrt{\nu}},$$

$$(47d) \quad \Phi_{4him} = a_{\alpha\beta} \Gamma_{\mu\sqrt{\pi}}^{\alpha} t_h^{\beta} t_i^{\pi} t_m^{\mu} - \Gamma_{h.i\sqrt{\nu}}.$$

Considering (41a, b) and (47a-d), we conclude that

$$(48a, b) \quad \Phi_{\frac{2}{1}him} = -\Phi_{\frac{1}{1}him}, \quad \Phi_{\frac{3}{1}him} = \Phi_{\frac{1}{1}him} + 2\Gamma_{h.i\check{v}}^h,$$

$$(48c) \quad \Phi_{\frac{4}{1}him} = -\Phi_{\frac{1}{1}him} - 2\Gamma_{h.i\check{v}}^h,$$

respectively

$$(48'a, b) \quad \Phi_{\frac{2}{1}im}^h = -\Phi_{\frac{1}{1}im}^h, \quad \Phi_{\frac{3}{1}im}^h = \Phi_{\frac{1}{1}im}^h + 2\Gamma_{i\check{v}}^h,$$

$$(48'c.) \quad \Phi_{\frac{4}{1}im}^h = -\Phi_{\frac{1}{1}im}^h - 2\Gamma_{i\check{v}}^h.$$

If me multiply equation (46) by $a_{\alpha\beta}N_{(\sigma)}^\beta$, we obtain

$$(49) \quad e_{(\sigma)}[\Omega_{\frac{1}{1}(\sigma)im} - \Omega_{(\sigma)im}] = a_{\alpha\beta}\Gamma_{\pi\check{v}}^\alpha N_{(\sigma)}^\beta t_m^\mu t_i^\pi, \quad \text{i.e.}$$

$$\Omega_{\frac{1}{1}(\sigma)i\check{v}} = e_{(\sigma)} a_{\alpha\beta}\Gamma_{\pi\check{v}}^\alpha N_{(\sigma)}^\beta t_m^\mu t_i^\pi.$$

By a direct ckeck based on (43b-d), or using (49), (42), we get

$$(49') \quad \Omega_{\frac{1}{1}(\sigma)i\check{v}} = \Omega_{(\sigma)m\check{v}} = \Omega_{\frac{3}{3}(\sigma)i\check{v}} = \Omega_{\frac{4}{4}(\sigma)m\check{v}} = e_{(\sigma)} a_{\alpha\beta}\Gamma_{\pi\check{v}}^\alpha N_{(\sigma)}^\beta t_i^\pi t_m^\mu.$$

From this

$$(50) \quad \begin{aligned} \Omega_{\frac{1}{1}(\sigma)im} &= \Omega_{\frac{3}{3}(\sigma)im} = \Omega_{\frac{1}{1}(\sigma)\underline{im}} + \Omega_{\frac{1}{1}(\sigma)i\check{v}} = \Omega_{(\sigma)im} + \Omega_{\frac{1}{1}(\sigma)i\check{v}}, \\ \Omega_{\frac{2}{2}(\sigma)im} &= \Omega_{\frac{4}{4}(\sigma)im} = \Omega_{\frac{1}{1}(\sigma)\underline{im}} + \Omega_{\frac{1}{1}(\sigma)i\check{v}} = \Omega_{(\sigma)im} - \Omega_{\frac{1}{1}(\sigma)i\check{v}}, \quad \text{i.e.} \\ \Omega_{\theta(\sigma)im} &= \Omega_{(\sigma)im} \pm \Omega_{\frac{1}{1}(\sigma)i\check{v}}, \end{aligned}$$

where for $\theta = 1, 3$ we take the sign $+$, and for $\theta = 2, 4$ the sign $-$, and $\Omega_{(\sigma)i\check{v}}$ is given by (49').

Further, from (28a, b)

$$(51a) \quad N_{(\sigma)\frac{1}{1}m}^\alpha = N_{(\sigma)\frac{3}{3}m}^\alpha = N_{(\sigma);m}^\alpha + \Gamma_{\pi\check{v}}^\alpha N_{(\sigma)}^\pi t_m^\mu,$$

$$(51b) \quad N_{(\sigma)\frac{2}{2}m}^\alpha = N_{(\sigma)\frac{4}{4}m}^\alpha = N_{(\sigma);m}^\alpha + \Gamma_{\mu\check{v}}^\alpha N_{(\sigma)}^\mu t_m^\mu,$$

and this

$$(52) \quad (N_{(\sigma)\frac{1}{1}m}^\alpha + N_{(\sigma)\frac{2}{2}m}^\alpha)/2 = (N_{(\sigma)\frac{3}{3}m}^\alpha + N_{(\sigma)\frac{4}{4}m}^\alpha)/2 = N_{(\sigma);m}^\alpha.$$

From here, considering (37') and (21b) we obtain (45) and also

$$(53) \quad (\Psi_{\frac{1}{1}(\varrho\sigma)m} + \Psi_{\frac{2}{2}(\varrho\sigma)m})/2 = (\Psi_{\frac{3}{3}(\varrho\sigma)m} + \Psi_{\frac{4}{4}(\varrho\sigma)m})/2 = \Psi_{\varrho\sigma m}.$$

If we put the covariant derivation into (51a) based on (37') and (21b), we get

$$(54) \quad -e_{(\sigma)} g^{ps} \Omega_{1(\sigma)s\check{\nu}} t_p^\alpha + \sum_p \left(\Psi_{1(\varrho\sigma)m} - \Psi_{(\varrho\sigma)m} \right) N_{(\varrho)}^\alpha = \Gamma_{\check{\nu}\mu}^\alpha N_{(\sigma)}^\pi t_m^\mu.$$

If we multiply this equation by $a_{\alpha\beta} t_i^\beta$, we get

$$(55) \quad \begin{aligned} -e_{(\sigma)} g^{ps} g_{p\check{\nu}} \Omega_{1(\sigma)s\check{\nu}} &= a_{\alpha\beta} \Gamma_{\check{\nu}\mu}^\alpha t_i^\beta t_m^\mu N_{(\sigma)}^\pi, \quad \text{i.e.} \\ \Omega_{1(\sigma)i\check{\nu}} &= e_{(\sigma)} a_{\alpha\beta} \Gamma_{\check{\nu}\mu}^\alpha t_i^\beta t_m^\mu N_{(\sigma)}^\pi, \end{aligned}$$

which is another form for (49). From here, in accordance with (42), we have

$$(55') \quad \Omega_{1(\sigma)i\check{\nu}} = \Omega_{2(\sigma)m\check{\nu}} = \Omega_{3(\sigma)i\check{\nu}} = \Omega_{4(\sigma)m\check{\nu}} = e_{(\sigma)} a_{\alpha\beta} \Gamma_{\check{\nu}\mu}^\alpha t_i^\beta t_m^\mu N_{(\varrho)}^\pi,$$

which is another form of (49').

Multiplying the equation (54) by $a_{\alpha\beta} N_{(\tau)}^\beta$, we get

$$(56) \quad e_{(\tau)} (\Psi_{1(\tau\sigma)m} - \Psi_{(\tau\sigma)m}) = a_{\alpha\beta} \Gamma_{\check{\nu}\mu}^\alpha N_{(\tau)}^\beta N_{(\sigma)}^\pi t_m^\mu,$$

respectively

$$(56') \quad \Psi_{1(\tau\sigma)m} = \Psi_{3(\tau\sigma)m} = \Psi_{(\tau\sigma)m} + e_{(\tau)} a_{\alpha\beta} \Gamma_{\check{\nu}\mu}^\alpha N_{(\tau)}^\beta N_{(\sigma)}^\pi t_m^\mu.$$

Analogously to (56), we get

$$(57) \quad e_{(\tau)} \left(\Psi_{2(\tau\sigma)m} - \Psi_{(\tau\sigma)m} \right) = a_{\alpha\beta} \Gamma_{\check{\nu}\mu}^\alpha N_{(\tau)}^\beta N_{(\sigma)}^\pi t_m^\mu, \quad \text{i.e.}$$

$$(57') \quad \Psi_{2(\tau\sigma)m} = \Psi_{4(\tau\sigma)m} = \Psi_{(\tau\sigma)m} - e_{(\tau)} a_{\alpha\beta} \Gamma_{\check{\nu}\mu}^\alpha N_{(\tau)}^\beta N_{(\varrho)}^\pi t_m^\mu.$$

From (56', 57') one gets (53). If we introduce a tensor

$$(58) \quad h_{(\tau\sigma)m} = e_{(\tau)} a_{\alpha\beta} \Gamma_{\check{\nu}\mu}^\alpha N_{(\tau)}^\beta N_{(\sigma)}^\pi t_m^\mu,$$

from (56', 57')

$$(59) \quad \Psi_{\theta(\tau\sigma)m} = \Psi_{(\tau\sigma)m} \pm h_{(\tau\sigma)m},$$

where for $\theta = 1, 3$ we take the sing + and for $\theta = 2, 4$ the sing -.

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Ekonomski fakultet
18000 Niš
Jugoslavija

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