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# ON REDUCED PRODUCTS OF KRIPKE MODELS

**Zoran Marković**

**Abstract**. Ultraproducts of Kripke models for Induitionistic theories were defined by Cleave [**1**] and Gabbay [**2**]. In [**3**] and [**4**] Gabbay proved the "Los's theorem" and some other analogues of classical results. Here we consider the products of Kripke models reduced over arbitrary filters, so called reduced products. Several classes of formulas are defined, for which preservation results are proved. Some preliminary results on this topic were contained in [**6**].

**Introduction**. We regard a Kripke model for a language $L$, classicaly, as a partially ordered set of classical structures $\mathfrak{A}_t$ for the language $L : \mathfrak{M} = \langle\langle T, \leq, 0\rangle; \mathfrak{A}_t : t \in T\rangle$ where $\langle T, \leq, 0\rangle$ is a partially ordered index set, with the least element 0, satisfying the condition: $s \leq t$ implies $\mathfrak{A}_s$ is a positive submodel of $\mathfrak{A}_t$. Forcing relation is defined as usual ([**5, 6**]).

Let $\mathfrak{M}_i = \langle\langle T_i, \leq_i, 0_i\rangle; \mathfrak{A}_t : t \in T_i\rangle (i \in I)$ be a set of Kripke models for a language $L$. We may assume $T_i \cap T_j = \varnothing$ for $i \neq j$. Let $F$ be a filter over $I$ and let $\Pi_F\langle T_i, \leq_i, 0_i\rangle = \langle T, \leq_F, 0_F\rangle$ be the reduced product of structures $\langle T_i, \leq_i, 0_i\rangle$. We denote the elements of $T$ by $\alpha_F, \beta_F, \ldots$ where $\alpha, \beta, \ldots \in \Pi_{i\in I}T_i$. Since the theory of partial order with the least element is a Horn theory, $\langle T, \leq_F, 0_F\rangle$ is a partially ordered set with the least element $0_F = \{\alpha \in \Pi_{i\in I}T_i : \{i : \alpha(i) = 0_i\} \in F\}$. If the context permits,shall avoid subscripts in $\leq_i$ and $\leq_F$. We define now the classical structure for $L$ associated with $\alpha_F \in T$. For $i \in I$ let $A_i = \cup_{t\in T_i} A_t$ and let $A = \Pi_{i\in I}A_i$. Elements of $A$ shall be denoted by $\xi, \eta, \ldots$. Let $\xi_F = \{\eta \in A : \{i : \xi(i) = \eta(i)\} \in F\}$. Now define $A_{\alpha F} = \{\xi_F : \{i : \xi(i) \in A_{\alpha(i)}\} \in F\}$ the universe of the clasical structure associated with $\alpha_F$. We obtain the structure $\mathfrak{A}_{\alpha F}$ by defining e.g., the interpretation of an $n$-ary relation symbol $R \in L$ as $R^{\alpha F} = \{\langle \xi^1_F, \ldots, \xi^n_F\rangle : \{i : \langle \xi^1(i), \ldots, \xi^n(i)\rangle \in R^{\alpha(i)}\} \in F\}$, and similarily for function and individual constant symbols. It is proved in [**6**] that all these definitions are unambiguous.

*Remark*. The more intuitive definition of $\mathfrak{A}_{\alpha F}$ as $\Pi_F\mathfrak{A}_{\alpha(i)}$ (as in [**3, 4**]) is not correct since in case $\alpha_F = \beta_F$ and $\alpha \neq \beta$ we have $\Pi_F A_{\alpha(i)} \neq \Pi_F A_{\beta(i)}$. However,

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each of these structures is isomorphic to $\mathfrak{U}_{\alpha F}$ as defined above, so hat this error does not have consequences for the results obtained in [**3**]. If $\beta \in \alpha_F$, the natural isomorphism of $\mathfrak{U}_{\alpha F}$ and $\Pi_F \mathfrak{U}_{\beta(i)}$ is defined by mapping any $\xi_F \in A_{\alpha F}$ to the set $\{\eta \in \Pi_{i\in I} A_{\beta(i)} : \{i : \xi(i) = \eta(i)\} \in F\}$.

**Results**. In the context of reduced products of models, the question of reduced product formulas and reduced factor formulas naturally arises. We shall consider first the class of formulas which hold in the reduced product iff they hold in filter-many factors of the product.

*Definition* 1. Let (RPF) be the class of formulas $\varphi(x_1, \ldots, x_n)$ satisfying the condition:

For any collection $\mathfrak{M}_i (i \in I)$ of Kripke structures for $\varphi$, any filter $F$ over $I$ and any $\alpha_F \in \Pi T_i$ and any $\xi^1_F, \ldots, \xi^n_F \in A_{\alpha F}$

$$\alpha \Vdash \varphi[\xi^1_F, \ldots, \xi^n_F] \quad \text{if} \quad \{i \in I : \alpha(i) \Vdash \varphi[\xi^1(i), \ldots, \xi^n(i)]\} \in F$$

In order to simplify the notation somewhat, from now on we shall suppress the valuation $\xi^1_F, \ldots, \xi^n_F$ (i.e. write $\varphi$ instead of $\varphi[\ldots]$). Unless explicitly stated, this does not imply that the formula in question is a sentence.

THEOREM 1. (i) (RPF) *contains atomic formulas* (ii) (RPF) *is closed under* $\wedge, \exists,$ *and* $\forall$.

*Proof*. (i) By definition of reduced products. (ii) Let $\varphi$ and $\psi$ be in (RPF) and let $\mathfrak{M}_i (i \in I)$ be Kripke structures for $\varphi$ and $\psi$ and let $F$ and $\alpha_F$ be as in Definition 1.

(a) $\alpha_F \Vdash \varphi \wedge \psi$ iff $\alpha_F \Vdash \varphi$ and $\alpha_F \Vdash \psi$ iff (since $\varphi,\ \psi \in$ (RPF))

$\{i : \alpha(i) \Vdash \varphi\} \in F$ and $\{i : \alpha(i) \Vdash \psi\} \in F$ iff $(\{i : \alpha(i) \Vdash \varphi \wedge \psi\} \in F$.

(b) Let $\alpha_F \Vdash \exists x \varphi(x)$. Then for some $\xi_F \in A_{\alpha F},\ \alpha_F \Vdash \varphi[\xi_F]$. Since $\varphi$ is an (RPF) formula, it follows that $\{i : \alpha(i) \Vdash \varphi[\xi(i)]\} \in F$. But $\alpha(i) \Vdash \varphi[\xi(i)]$ implies $\alpha(i) \Vdash \exists x \varphi(x)$, so $\{i : \alpha(i) \Vdash \exists x \varphi(x)\} \in F$.

Conversely, let $X = \{i : \alpha(i) \Vdash \exists x \varphi(x)\} \in F$. Then for $i \in X$, there exists $a_i \in A_{\alpha(i)}$ such that $\alpha(i) \Vdash \varphi[a - i]$. Let $\xi$ be an element of $\Pi_{i\in I} A_{\alpha(i)}$ such that for $i \in X,\ \xi(i) = a_i$. Then $\xi_F \in A_{\alpha F}$ and $\{i : \alpha(i) \Vdash \varphi[\xi(i)]\} \supseteq X \in F$. Since $\varphi$ is an (RPF) formula, it follows that $\alpha_F \Vdash \varphi[\xi_f]$ and so $\alpha_F \Vdash \exists x \varphi(x)$.

(c) Suppose $X = \{i : \alpha(i) \Vdash \forall x \varphi(x)\} \notin F$. Then for $i \in I - X$ there exists an $s_i \in T_i$ such that $\alpha(i) \le s_i$ and there exists an $a_i \in A_{si}$ such that it is not case that $s_i \Vdash \varphi[a_i]$. Let for $i \in I$,

$$\beta(i) = \begin{cases} s_i & \text{iff } i \notin X \\ \alpha(i) & \text{iff } i \in X \end{cases} \quad \text{and,} \quad \xi(i) = \begin{cases} a_i & \text{for } i \notin X \\ a & \text{for } i \in X \end{cases}$$

where a is an arbitrary element of $A_{\alpha(i)}$.

Now clearly for every $i \in I$ we have $\alpha(i) \leq \beta(i)$ and $\xi(i) \in A_{\beta(i)}$, thus $\alpha_F \leq \beta_F$ and $\xi_F \in A_{\beta F}$. But $\{i : \beta(i) \Vdash \varphi[\xi(i)]\} \subseteq X \notin F$ so not $\beta_F \Vdash \varphi[\xi_F]$ since $\varphi$ is an (RPF) formula. Therefore not $\alpha_F \Vdash \forall x\varphi(x)$.

Conversely, assume $X = \{i : \alpha(i) \Vdash \forall x\varphi(x)\} \in F$ and let $\alpha_F \leq \beta_F$ and $\xi_F \in A_{\beta F}$. This means that $X_\beta = \{i : \alpha(i) \leq \beta(i)\} \in F$ and $X_\xi = \{i : \xi(i) \in A_{\beta(i)}\} \in F$. Then $Z = X \cap X_\beta \cap X_\xi \in F$. Now $i \in Z$ implies $\beta(i) \Vdash \varphi[\xi(i)]$ so $\{i : \beta(i) \Vdash \varphi[\xi(i)]\} \in F$. As $\varphi$ is an (RPF) formula this implies $\beta_F \Vdash \varphi[\xi_F]$. Since $\beta_F$ and $\xi_F$ were arbitrary, it follows that $\alpha_F \Vdash \forall x\varphi(x)$.

*Definition* 2. Let (RF) (Reduced Factor formulas) be the class of all formulas $\varphi(x_1, \ldots, x_n)$ satisfying the following condition:

For any collection $\mathfrak{M}_i (i \in I)$ of Kripke structures for $\varphi$, any filter $F$ over $I$, any $\alpha_F \in \Pi_F T_i$ and any $\xi_F^1, \ldots, \xi_F^n \in A_{\alpha F}$.

$$\alpha_F \Vdash \varphi[\xi_F^1, \ldots, \xi_F^n] \text{ implies } \{i \in I : \alpha(i) \Vdash \varphi[\xi^1(i), \ldots, \xi^n(i)]\} \in F$$

THEOREM 2. (i) (RF) *contains* (RPF) (ii) (RF) *is closed under* $\vee$, $\wedge$, $\exists$ *and* $\forall$.

*Proof*. (i) Obvious (ii) The proofs for $\wedge, \exists$ and $\forall$ are practically the same as the first halves of (a), (b) and (c) in the proof of Theorem 1 (ii).

Assume $\alpha_F \Vdash \varphi \vee \psi$. Then $\alpha_F \Vdash \varphi$ or $\alpha_F \Vdash \psi$. If $\varphi$ and $\psi$ are in (RF) it follows that $\{i : \alpha(i) \Vdash \varphi\} \in F$ or $\{i : \alpha(i) \Vdash \psi\} \in F$. But, the set $\{i : \alpha(i) \Vdash \varphi \vee \psi\}$ contains both, so it also is in $F$.

We shall need the following result ([**6**, Lemma III 2, 3]).

LEMMA 3. *Assume that* $\{i : \alpha(i) \Vdash \neg\varphi \vee \psi\} \in F$ *and that* $\varphi$ *is a reduced factor formula. Then either* $\alpha_F \Vdash \neg\varphi$ *or* $\{i : \alpha(i) \Vdash \psi\} \in F$.

*Proof*. Suppose that not $\alpha_F \Vdash \neg\varphi$. Then for some $\beta_F \geq \alpha_F$, $\beta_F \Vdash \varphi$. Let $X = \{i : \alpha(i) \Vdash \neg\varphi \vee \psi\} \in F$, $X_\beta = \{i : \alpha(i)\} \in F$. Since $\varphi$ is a reduced factor formula, we have $Z = \{i : \beta(i) \Vdash \varphi\} \in F$. Let $U = \{i : \alpha(i) \Vdash \neg\varphi\}$ and $V = \{i : \alpha \Vdash \psi\}$. Clearly $U \cup V \in F$. Also $X_\beta \cap Z \in F$. Then $(X_\beta \cap Z) \cap (U \cup V) \in F$. However $X_\beta \cap Z \cap U = \varnothing$, so we must have $X_\beta \cap Z \cap V \in F$. Therefore $V \in F$.

*Definition* 3. Let (RP) (Reduced product formulas) be the class of all formulas $\varphi(x_1, \ldots, x_n)$ satisfying the following condition:

For any collection $\mathfrak{M}_i (i \in I)$ of Kripke structures for $\varphi$, any filter $F$ over $I$, any $\alpha_F \in \Pi_F T_i$ and any $\xi_F^1, \ldots, \xi_F^n \in A_{\alpha F}$.

$$\{i : \alpha(i) \Vdash \varphi[\xi^1(i), \ldots, \xi^n(i)]\} \in F \quad \text{implies} \quad \alpha_F \Vdash \varphi[\xi_F^1, \ldots, \xi_F^n]$$

THEOREM 4. (i) (RP) *contains* (RPF)

(ii) *If* $\varphi \in$ (RF) *and* $\psi \in$ (RP) *then* (a) $(\varphi \to \psi) \in$ (RP)

(b) $\rceil\varphi \in$ (RP)

(c) $(\rceil\varphi \vee \psi) \in$ (RP)

(iii) (RP) *is closed under* $\wedge, \exists$ *and* $\forall$.

*Proof*. (i) Obvious. (ii) (a) Let $X = \{i : \alpha(i) \Vdash \varphi \to \psi\} \in F$. We have to show that for any $\beta_F \geq \alpha_F$, if $\beta_F \Vdash \varphi$ then $\beta_F \Vdash \psi$. Assume $\alpha_F \leq \beta_F$ and $\beta_F \Vdash \varphi$ and let $X_\beta = \{i : \alpha(i) \leq \beta(i)\} \in F$ Since $\varphi$ is a reduced factor formula, we have $X_\varphi = \{i : \beta(i) \Vdash \varphi\} \in F$. Then $Z = X \cap X_\beta \cap X_\varphi \in F$. But $i \in Z$ implies $\beta(i) \Vdash \psi$, so $\{i : \beta(i) \Vdash \psi\} \supseteq Z \in F$. As $\psi$ is assumed to be a reduced product formula, it follows that $\beta_F \Vdash \psi$.

(b) $\rceil\varphi$ is defined as $\varphi \to \Box$ where $\Box$ (absurdity) is an atomic formula, so this is a special case of (a).

(c) Let $X = \{i : \alpha(i) \Vdash \rceil\varphi \vee \psi\} \in F$. As $\varphi$ is a reduced factor formula, using Lemma 3. we get that $\alpha_F \Vdash \rceil\varphi$ or $\{i : \alpha(i) \Vdash \psi\} \in F$. In the latter case we have $\alpha_F \Vdash \psi$, as $\psi$ is a reduced product formula.

(iii) The proofs are practically the sane as second halves od (a) (b) and (c) in the proof of Theorem 1. (ii).

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Matematički Institut
Knez-Mihailova 35
11000 Beograd
Jugoslavija

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