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# LINEAR COMBINATIONS OF REGULAR FUNCTIONS WITH NEGATIVE COEFFICIENTS

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**Abstract**. Let $f(z) = a_p z^p - \sum_{n=1}^{\infty} a_{n+k} z^{n+k}$, $k \geq p \geq 1$ with $a_p > 0$, $a_{n+k} \geq 0$ be regular in $E = \{z : |z| < 1\}$ and $F(z) = (1-\lambda)f(z) + \lambda z f'(z)$, $z \in E$ where $\lambda \geq 0$. The radius of $p$-valent starlikeness of order $\alpha, 0 \leq \alpha < 1$, of $F$ as $f$ varies over a certain subclass of $p$-valent regular functions in $E$ is determined, and the mapping properties of $F$ in certain other situations also are discussed.

**1. Introduction**. Let $E = \{z :| z |< 1\}$ be the unit disc in $\mathbf{C}$ and $H = \{w : w$ is regular in $E$ such that $w(0) = 0$, $| w(z) |< 1$, $z \in E\}$. Let $P(A, B)$ denote the class of functions regular in $E$ which are of the form $(1 + Aw(z))/(1 + Bw(z))$, $-1 \leq A < B \leq 1$, $w \in H$. Let $T_p$ be the class of functions $f(z) = a_p z^p - \sum_{n=1}^{\infty} a_{n+k} z^{n+k}$, $k \geq p \geq 1$, $a_p > 0$ and $a_{n+k} \geq 0$, regular in the unit disc $E$. Let $S_p^*(A, B) = \left\{ f \in T: \frac{1}{p} \frac{zf'(z)}{f(z)} \in P(A, B) \right\}$ and $K_p(A, B) = \left\{ f \in T_p : \frac{1}{p} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \in P(A, B) \right\}$. We note that $S_p^*(A, B)$ and $K_p(A, B)$ are subclasses of $T_p$ consisting of $p$-valently starlike and $p$-valently convex functions respectively. $f \in S_p^*(A, B)$ implies that, $Re\{zf'(z)/f(z)\} > 0$ for $z \in E$. Further if $f \in S_p^*(A, B)$ and $z = re^{i\theta}$, $r < 1$, $\frac{1}{2\pi} \int_0^{2\pi} Re \frac{zf'(z)}{f(z)} d\theta = \frac{p}{2\pi} \int_0^{2\pi} Re \frac{1+Aw(z)}{1+Bw(z)} d\theta = p$, since $Re \frac{1+Aw(z)}{1+Bw(z)}$ is a harmonic function in $E$ with $w(0) = 0$. This argument shows the $p$-valence of $f$ in $S_p^*(A, B)$. Similarly $f \in K_p(A, B)$ is p-valently convex in $E$. Define $P^*(A, B) = \{f \in T_1 : f'(z) \in P(A, B), a_1 = 1\}$. In this paper we consider the function $F$ in $E$ defined by $F(z) = (1-\lambda)f(z) + \lambda z f'(z)$, $\lambda \geq 0$ and study some mapping properties of $F$, as $f$ varies over the classes $S_p^*(A, B)$, $K_p(B, A)$ and $P^*(A, B)$. We also consider the class of function $f \in T_p$ for which $f^{(p-1)}$ is univalent and discuss a mapping property of $F$. It is interesting to note that the necessary condition for univalence of $f^{(p-1)}$ for $f \in T_p$, turns out to be a necessary and sufficient condition for $f^{(p-1)}$ to be starlike univalent in $E$.

**2. Coefficient inequalities and theorems on radius of starlikeness, convexity and close-to-convexity**. We use the following notations for the sake of brevity

$$n+k=m,\ \ m(B+1)-p(A+1)=C_m,\ \ p(B-A)=D \ \text{ and } \ \sum_{m=k+1}^{\infty}=\Sigma.$$

We begin by proving the following

LEMMA 1. *Let $f \in T_p$. Then $f \in S_p^*(A,B)$ if and only if*

$$\Sigma C_m a_m \le D a_p. \tag{1}$$

*Proof*. Suppose $f \in S_p^*(A,B)$. Then $\frac{zf'(z)}{f(z)} = p\frac{1+Aw(z)}{1+Bw(z)}$, $-1 \le A < B \le 1$, $w \in H$.
That is,

$$w(z)=\frac{p-zf'(z)/f(z)}{Bzf'(z)/f(z)-Ap},\ w(0)=0 \quad\text{and}$$
$$|\, w(x)\,| = \left|\frac{zf'(z)-pf(z)}{Bzf'(z)-Apf(z)}\right| = \left|\frac{\Sigma(m-p)a_m z^m}{Da_p z^p-\Sigma(Bm-Ap)a_m z^m}\right| < 1.$$

Thus

$$\mathrm{Re}\left\{\frac{\Sigma(m-p)a_m z^m}{Da_p z^p \Sigma(Bm-Ap)a_m z^m}\right\} < 1. \tag{2}$$

Take $z=r$ with $0<r<1$. Then, for sufficiently small $r$, the denominator of the left hand member of (2) is positive and so it is positive for all $r$ $0<r<1$, since $w(z)$ is regular for $|\, z\,|<1$. Then (2) gives

$$\Sigma(m-p)a_m r^m < Da_p r^p - \Sigma(Bm-Ap)a_m r^m,\ \text{ that is,}$$
$$\Sigma[m(B+1)-p(A+1)]a_m r^m < Da_p r^p,\ \text{ that is, }\ \Sigma C_m a_m r^m < Da_p r^p,$$

and (1) follows on letting $r \to 1$.

Conversely, for $|\, z\,| = r$, $0<r<1$, and since $r^m < r^p$, by (1) we have $\Sigma C_m a_m r^m < r^p \Sigma C_m a_m < Da_p r^p$. Using this inequality we have

$$|\, \Sigma(m-p)a_m z^m \,| \le \Sigma(m-p)a_m r^m < Da_p r^p - \Sigma(Bm-Ap)a_m r^m$$
$$\le |\, Da_p z^p - \Sigma(Bm-Ap)a_m z^m \,|\,.$$

This proves that $zf'(z)/f(z)$ is on the form $p\frac{1+Aw(z)}{1+Bw(z)}$ with $w \in H$. Therefore $f \in S_p^*(A,B)$ and the proof is complete.

Corollary 1. *$f \in T_1$. Then $f \in S_1^*(2\alpha-1,1)$ if and only if* $\Sigma(m-\alpha)a_m \leq (1-\alpha)a_1$.

*Remark*. Note that $S_1^*(2\alpha-1,\ 1) = S^*(\alpha)$, the class of univalent starlike functions of order $\alpha$. This corollary reduces to Theorem 1 in [**2**].

Theorem 1. *Let $f \in S_p^*(A,B)$ and $F(z) = (1-\lambda)f(z)+\lambda zf'(z)$, $\lambda \geq 0$, $z \in E$. Then $F$ is $p$-valently starlike of order $\alpha$, $0 \leq \alpha < 1$, for*

$$\mid z \mid < r_1 = \inf_m \left[\frac{(p-\alpha)(1+(p-1))}{(m-\alpha)(1+(m-1))}\cdot\frac{C_m}{D}\right]^{1/(m-p)} \quad \textit{and the bound is sharp.}$$

*Proof*. Since $F(z) = (1-\lambda)f(z)+\lambda zf'(z),\ \ \lambda \geq 0$

$$= (1+(p-1)\lambda)a_p z^p - \Sigma(1+(m-1)\lambda)a_m z^m,$$

$$\frac{zF'(z)}{F(z)} = \frac{p(1+(p-1)\lambda)a_p z^p - \Sigma m(1+(m-1)\lambda)a_m z^m}{(1+(p-1)\lambda)a_p z^p - \Sigma(1+(m-1)\lambda)a_m z^m}.$$

It suffices to show that $\mid zF'(z)/F(z)-p \mid \leq p-\alpha$ for $\mid z \mid < r_1$. Now

$$(3)\qquad \begin{aligned}\left|\frac{zF'(z)}{F(z)}-p\right| &= \left|\frac{-\Sigma(m-p)(1+(m-1)\lambda)a_m z^m}{(1+(p-1)\lambda)a_p z^p - \Sigma(1+(m-1)\lambda)a_m z^m}\right| \\ &\leq \frac{\Sigma(m-p)(1+(m-1)\lambda)a_m \mid z\mid^{m-p}}{\mid (1+(p-1)\lambda)a_p - \Sigma(1+(m-1)\lambda)a_m \mid z \mid^{m-p}\mid}.\end{aligned}$$

Consider the values of $z$ for which $\mid z \mid < r_1$, so that

$$\mid z \mid^{m-p} \leq \frac{(p-\alpha)(1+(p-1)\lambda)}{(m-\alpha)(1+(m-1)\lambda)}\cdot\frac{C_m}{D} \ \text{ holds. Then}$$

$$\begin{aligned}\Sigma\frac{1+(m-1)\lambda}{1+(p-1)\lambda}\cdot a_m \mid z\mid^{m-p} &\leq \Sigma\frac{(p-\alpha)}{(m-\alpha)}\cdot\frac{C_m}{D}a_m \\ &\leq \Sigma\frac{C_m}{D}a_m < a_p, \ \text{ which is true by (1).}\end{aligned}$$

Thus, the expression within the modulus sign in the denominator of the right hand side of (3), for the considered values of $z$ is positive and so we have

$$\left|\frac{zF'(z)}{(Fz)}-p\right| \leq \frac{\Sigma(m-p)(1+(m-1)\lambda)a_m \mid z \mid^{m-p}}{(1+(p-1)\lambda)a_p - \Sigma(1+(m-1)\lambda)a_p \mid z\mid^{m-p}} \leq p-\alpha \ \text{ if}$$

$$\Sigma(m-\alpha)(1+(m-1)\lambda)a_m \mid z \mid^{m-p} \leq (p-\alpha)(1+(p-1)\lambda)a_p,$$

that is, if

$$(4)\qquad \Sigma\frac{(m-\alpha)(1+(m-1)\lambda)a_m \mid z\mid^{m-p}}{(p-\alpha)(1+(p-1)\lambda)a_p} \leq 1.$$

By Lemma 1 we have $f \in S_p^*(A,B)$ if and only if $\Sigma \frac{C_m a_m}{D a_p} \leq 1$. Hence (4) is true if

$$\frac{(m-\alpha)(1+(m-1)\lambda)}{(p-\alpha)(1+(p-1)\lambda)} \mid z \mid^{m-p} \leq \frac{C_m}{D},$$

that is, if

$$\mid z \mid \leq \left[\frac{(p-\alpha)(1+(p-1)\lambda)}{(m-\alpha)(1+(m-1)\lambda)} \cdot \frac{C_m}{D}\right]^{1/(m-p)}.$$

To see the $p$-valence of $F$ in $\mid z \mid < r_1$, we observe that $zF'(z)/F(z)$ is regular in $\mid z \mid < r_1$ and hence $\mathrm{Re}(zF'(z)/F(z))$ is harmonic in that disc. For $r < r_1$ and $z = re^{i\theta}$, $\frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\frac{zF'(z)}{F(z)} d\theta = p$, showing that $F$ is $p$-valent in $\mid z \mid < r_1$.

Hence the proof follows. The extremal function is given by $f(z) = a_p z^p - (Da_p/C_m)z^m$.

*Remark.* We obtain Theorem 2 in [**1**] as a particular case.

COROLLARY 2. *If $f \in S_p^*(A,B)$, then $f$ is $p$-valently starlike of order $\alpha$, $0 \leq \alpha < 1$, in*

$$\mid z \mid < \inf_m \left[\frac{(p-\alpha)}{(m-\alpha)} \cdot \frac{C_m}{D}\right]^{1/(m-p)}, \ m = k+1, k+2, \dots .$$

*Proof.* Put $\lambda = 0$ in Theorem 1.

COROLLARY 3. *If $f \in S_p^*(A,B)$, then $f$ is $p$-valently convex of order $\beta$, $0 \leq \beta < 1$, in*

$$\mid z \mid < \inf_m \left[\frac{p}{m} \cdot \frac{(p-\beta)}{(m-\beta)} \cdot \frac{C_m}{D}\right]^{1/(m-p)}, \ m = k+1, \dots .$$

*Proof.* Put $\lambda = 1$ in Theorem 1 and note that $zf'(z)$ is starlike of order $\beta$ if and only if $f(z)$ is convex of order $\beta$.

*Remark.* For $k = p = 1,\ A = 2\alpha - 1,\ B = 1,\ \beta = 0$, Corollary 3 reduces to Theorem 8 in [**2**].

COROLLARY 4. *If $f \in S_p^*(A,B)$ and $F(z) = \frac{(z^c f(z))'}{1+c}$, $c = 1, 2, \dots, z \in E$, then $F$ is $p$-valently starlike of order $\alpha$, $0 \leq \alpha < 1$, in*

$$\mid z \mid \leq \inf_m \left[\frac{(p-\alpha)(p+c)}{(m-\alpha)(m+p)} \cdot \frac{C_m}{D}\right]^{1/(m-p)}.$$

*Proof.* Put $\lambda = 1/(c+1)$ in Theorem 1.

THEOREM 2. *Let $f \in K_p(A,B)$ and $F(z) = (1-\lambda)f(z) + \lambda z f'(z)$, $\lambda \geq 0$, $z \in E$. Then $\frac{(F'/d')}{(1+(p-1)\lambda)} \in P(A_1, B)$, where $A_1 = \frac{A\lambda p + (1-\lambda)B}{1+(p-1)\lambda}$, if $\lambda < \frac{1+B}{1+B-p(A+1)}$ and*

*$p < \frac{1+B}{1+A}$. In particular $F$ is p-valently close-to-convex in $E$ if $\lambda < \frac{1+B}{1+B-p(A+1)}$ and $p < \frac{1+B}{1+A}$. Also $F$ is p-valently convex of order $\alpha$, $0 \le \alpha < 1$, in $|\, z \,| < r_1$. This bound is sharp.*

*Proof*. Since $f \in K_p(A,B)$. We have

$$\frac{F'(z)}{f'(z)} = (1-\lambda) + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) = 1 - \lambda + \lambda p \frac{1+Aw(z)}{1+Bw(z)}$$
$$= \frac{(1+(p-1)\lambda) + (A\lambda p + B(1-\lambda))w(z)}{1+Bw(z)}, \quad \text{where } w \in H.$$

Therefore

$$\frac{1}{(1+(p-1)\lambda)}\frac{F'(z)}{f'(z)} = \frac{1+A_1 w(z)}{1+Bw(z)} \quad \text{where } A_1 = \frac{A\lambda + p + B(1-\lambda)}{1+\lambda(p-1)}.$$

Evidently $A_1 < B$ since $A < B$ and $A_1 > -1$, provided $\lambda < \frac{B+1}{1+B-p(A+1)}$ and $p < \frac{1+B}{1+A}$. Therefore, $\frac{(F'/f')}{1+(p-1)\lambda} \cdot \in P(A_1, B)$ with $-1 < A_1 < B \le 1$. In particular $\operatorname{Re} F'/f' > 0$. Further since $f$ is $p$-valently convex, it follows that $F$ is $p$-valent by a theorem of Umezewa [**3**, Theorem 1]. It follows that $F$, if $p$-valently close-to-convex in $E$ if $\lambda < \frac{B+1}{B+1-p(A+1)}$ and $p < \frac{B+1}{A+1}$. We now prove that $F(z)$ is convex of order $\alpha$, $0 \le \alpha < 1$, in $|\, z \,| < r_1$ where $r_1$ is as given in Theorem 1. $zF'(z) = (1-\lambda)zf'(z) + \lambda z(zf'(z))'$ for $z \in E$. $f \in K_p(A,B)$ if and only if $zf' \in S_p^*(A,B)$. Applying Theorem 1 to $zf'$ we conclude that $zF'(z)$ is $p$-valently starlike of order $\alpha$ for $|\, z \,| < r_1$. So, $F$ is $p$-valently convex of order $\alpha$ for $|\, z \,| < r_1$. The extremal function is given by $f(z) = a_p z^p - (Dpa_p/mC_m)z^m$.

*Remark*. We get Theorem 3 in [**1**] as a special case.

Lemma 2. *Let $f \in T_1$, $a_1 = 1$. Then $f \in P^*(A,B)$ if and only if*

$$\Sigma m(B+1)a_m \le B - A. \tag{5}$$

*Proof*. Proof of Lemma 2 is similar to the proof of Lemma 1 and is hence omitted.

Theorem 3. *Let $f \in P^*(A,B)$ and $F(z) = (1-\lambda)f(z) + \lambda zf'(z)$ for $z \in E$, $\lambda \ge 0$. Then*

$$\operatorname{Re} F'(z) > \beta,\ 0 \le \beta < 1, \quad \textit{for} \quad |\, z \,| < r_2 = \inf_m \left[\frac{(1-\beta)}{1+(m-1)\lambda}\frac{B+1}{B-A}\right]^{1/(m-1)}$$

*and the bound is sharp.*

*Proof*. It suffices to show that $|\, F'(z) - 1 \,| \le 1 - \beta$ for $|\, z \,| < r_2$. Since $f \in P^*(A,B)$, using Lemma 2, we see that (5) holds. Since $F'(z) = 1 - \Sigma m(1 +$

$(m-1)\lambda)a_m z^{m-1}$, using (5), we see that $\mid F'(z)-1 \mid \leq \Sigma m(1+(m-1)\lambda)a_m \mid z \mid^{m-1} \leq 1-\beta$ provided

$$\frac{m(1+(m-1)\lambda)}{1-\beta} \mid z \mid^{m-1} \leq \frac{m(B+1)}{B-A}. \tag{6}$$

Now (6) holds if $\mid z \mid \leq \left[\frac{(1-\beta)}{1+(m-1)\lambda}\cdot\frac{B+1}{B-A}\right]^{1/(m-1)}$, and the proof follows.

The extremal function is $f(z) = z - \frac{B-A}{m(B+1)} z^m$.

*Remark*. Theorem 4 in [**1**] arises as a special case.

LEMMA 3. *Let $f \in T_p$ and let $f^{(p-1)}$ be univalent in $E$. Then*

$$\Sigma[m(m-1)\dots(m-p+1)]a_m \leq p!\, a_p. \tag{7}$$

*Proof*. Suppose $\Sigma[m(m-1)\dots(m-p+1)]a_m = p!a_p + \varepsilon,\ \varepsilon > 0$. Then there exists a positive integer $N > p$ such that

$$\sum_{m=k+1}^{N} [m(m-1)\dots(m-p+1)]a_m > p!a_p + \varepsilon/2. \tag{8}$$

Let $\left(\frac{p!a_p}{p!a_p+\varepsilon/2}\right)^{1/(N-p)} < z < 1$, so that $z^{N-p} > \frac{p!a_p}{p!a_p+\varepsilon/2}$. Also $z^{m-p} > z^{N-p}$ for $m < N$, since $z < 1$. Using these two inequalities and (8), we have

$$\begin{aligned} f^{(p)}(z) &= p!a_p - \Sigma[m(m-1)\dots(m-p+1)]a_m z^{m-p} \\ &\leq p!a_p - \sum_{m=k+1}^{N}[m(m-1)\dots(m-p+1)]a_m z^{m-p} \\ &\leq p!a_p - z^{N-p}\sum_{m=k+1}^{N}[m(m-1)\dots(m-p+1)]a_m \\ &< p!a_p - (p!a_p+\varepsilon/2)z^{N-p} < 0, \end{aligned}$$

and $f^{(p)}(0) = p!a_p > 0$. Therefore, there exists a point $z_0$ with $0 < z_0 < \left(\frac{p!a_p}{p!a_p+\varepsilon/2}\right)^{1/(N-p)} < 1$ such that $f^{(p)}(z_0) = 0$. This contradicts the univalence of $f^{(p-1)}(z)$ for $z \in E$. Hence the lemma.

*Remarks* 1. Putting $p = 1,\ a_p = 1$, this lemma reduces to Theorem 3 in [**2**].

2. Applying Corollary 1 with $\alpha = 0$ to $f^{(p-1)}(z)$ which belongs to the class $T_1$, we see that condition (7) is necessary and sufficient for $f^{(p-1)}$ to be starlike univalent.

THEOREM 4. *Let* $f \in T_p$ *and let* $f^{(p-1)}$ *be univalent in* $E$. *Suppose* $F(z) = (1-\lambda)f(z) + \lambda z f'(z)$, $\lambda \geq 0$, $z \in E$. *Then* $\mathrm{Re}\{zF'(z)/F(z)\} < \alpha$, $0 \leq \alpha < 1$, *for*

$$\mid z \mid < r_3 = \inf_m \left[ \frac{(p-\alpha)[1+(p-1)\lambda]}{(m-\alpha)[1+(m-1)\lambda]} \frac{m(m-1)\dots(m-p+1)}{p!} \right]^{1/(m-p)}$$

*and the bound is sharp.*

*Proof*. Let $F(z) = (1-\lambda)f(z) + \lambda z f'(z)$. Then inequality (3) holds. Now, since $f \in T_p$, by Lemma 3 inequality (7) holds. Consider the values of $z$ for which $\mid z \mid < r_3$ so that $\mid z \mid^{m-p} \leq \frac{(p-\alpha)(1+(p-1)\alpha)}{(m-\alpha)(1+(m-1)\lambda)} \cdot \frac{m(m-1)\dots(m-p+1)}{p!}$.
Then

$$\begin{aligned} \Sigma \frac{1+(m-1)\lambda}{1+(p-1)\lambda} a_m \mid z \mid^{m-p} &\leq \Sigma \frac{(p-\alpha)}{(m-\alpha)} \frac{m(m-1)\dots(m-p+1)}{p!} a_m \\ &< \Sigma \frac{m(m-1)\dots(m-p+1)}{p!} a_m \\ &< a_p, \quad \text{by}(7). \end{aligned}$$

Therefore as in the proof of Theorem 1, we can write $\mid zF'(z)/F(z) - p \mid \leq p - \alpha$ provided (4) holds. Using (7), we see inequality (4) also holds provided

$$\frac{(m-\alpha)(1+(m-1)\lambda)}{(p-\alpha)(1+(p-1)\lambda)} \mid z \mid^{m-p} \leq \frac{m(m-1)\dots(m-p+1)}{p!}, \text{ that is, if}$$

$$\mid z \mid \leq \left[ \frac{(p-\alpha)(1+(p-1)\lambda)}{(m-\alpha)(1+(m-1)\lambda)} \cdot \frac{m(m-1)\dots(m-p+1)}{p!} \right]^{1/(m-p)}.$$

The proof follows. The extremal function is given by

$$f(z) = a_p z^p - \frac{p! a_p}{m(m-1)\dots(m-p+1)} z^m.$$

It is easy to verify the univalence of $f^{(p-1)}(z)$ in $E$.

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