CONDITIONS FOR THE INTEGRABILITY OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATION, II

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Abstract. Conditions for the integrability of second order nonlinear differential equation (0.1) are derived. The obtained result contains, as a special cases a number of known results.

0. In this paper we continue the investigation of integrability of non- linear second order differential equations. We consider the following equation:

$$(0.1) \qquad y'' + P(y)y'^2 + Q(x,y)y' + R(x,y) = 0,$$

where P, Q, R are given functions.

Equations of this type were considered in a number of papers (see [1-16]). Only in Kamke's collection [1] 103 equations of the form (0.1) are noted. Also, Painlevé [2] considered many equations of the same form.

We purpose a method for solving equation (0.1) by reducing it to an equation of the form:

$$(0.2) \qquad Y'' + A(Y,x)Y' + B(Y,x) = 0,$$

which is considered in [5]. In [5] was proved that the equation (0.2) can be reduced to the autonomous equation

$$(0.3) \qquad d^2z/dt^2 + f(z)dz/dt + g(z) = 0$$

by means of transformations

$$(0.4) \qquad y = q(x)z(t) + r(x), \quad dt = p(x)dx,$$

if the following conditions are fulfilled:

$$(0.5) \qquad A(Y,x) = pf((Y-r)/q) - 2q'/q - p'/p,$$

$$B(Y,x) = p^2qg((Y-r)/q) - (q''/q + A(Y,x)q'/q)(Y-r) - (r'' + A(Y,x)r')$$

(r, g are some functions depending of one variable, p is a differentiable function of x; q, r are twice differentiable functions of x).

In Section 1 we shall derive the conditions for the integrability of (0.1) by using the above result from [**5**]. Also, some remarks and examples are given.

A certain special class of equations of the form (0.1) is treated in Section 2. Some particular cases are given and compared with some known results from [**7-9**], [**13-17**].

1. Substituting, in (0.1)

$$(1.1) \qquad Y = F(y),$$

where

$$(1.2) \qquad F(y) = K(\int_{y_0}^{y} \exp(\int_{y_0}^{y} P(y)dy)dy + L)$$

(K, L are constants) we obtain (0.2), where the functions A, B are given by:

$$A(K(\int_{y_0}^{y} \exp(\int_{y_0}^{y} P(y)dy)dy + L), x) = Q(x, y),$$

$$(1.3) \qquad B(K\int_{y_0}^{y} \exp(\int_{y_0}^{y} P(y)dy)dy + L), x) = K \exp(\int_{y_0}^{y} (P(y)dy R(x, y)$$

This means that the equations (0.1) and (0.2) are equivalent.

Furthermore, using (0.3) — (0.5) and (1.1) — (1.3), we conclude that (0.1) can be reduced to the autonomous form if the functions Q, R are given by:

$$(1.4) \qquad Q(x, y) = pf((F(y) - r)/q) - 2q'/q - p'/p,$$

$$R(x, y) = p^2 qg((F(y) - r)/q)/F'(y) - (q''/q + Q(x, y)q'/q)(F(y) - r)) - $$
$$- (r'' + Q(x, y)r').$$

In this case under the transformation

$$(1.5) \qquad F(y) = q(x)z(t) + r(x), \quad dt = p(x)dx,$$

where F is given by (1.2), the equation (0.1) reduces to the autonomous equation (0.3).

Then we obtain that the general solution of (0.1) is given by:

$$(1.6) \qquad \int (U(F(y) - r)/q, \ C)^{-1} d((F(y) - r)/q) = \int p(x)dx + D,$$

(C, D) are arbitrary constants), $u = U(z, C)$ is the general solution of the first order equation

$$(1.7) \qquad u(z)du/dz + f(z)u(z) + g(z) = 0$$

Remarks and *examples*. Here are some remarks and examples related to the above result.

1^0 Equation

$$(1.8) \qquad y'' + P(y)y'^2 + a(x)y' + b(x)T(y) = 0,$$

under the transformation $(1.1) — (1.2)$ reduces to the

$$Y'' + a(x)Y' + b(x)S(Y) = 0$$

where $S(F(y)) = F'(y)T(y)$. If $b(x) = k \exp(-2 \int a(x)dx)(k =$const) then the above equation is integrable (see e.g. [5, 10, 11]), so in this case, (1.8) is also integrable. This result is also obtained by J. D. Kečkić [10] and by L. M. Berkovič and N. N. Rozov [11].

2^0 In Kamke's collection [1] 103 equations of the type 0.1 are noted. These are equations $(6.45)—(6.53)$, (6.86), $(6.107),(6.109)—(6.131)$, $(6.137)—(6.153)$, $(6.155)—(6.176)$, (6.179), (6.180), $(6.183)—(6.186)$, $(6.189)—(6.192)$, $(6.196)—(6.204)$, (6.206), (6.207), (6.210), $(6.212)— (6.215)$, $(6.222)—(6.224)$. Many of these equations can be solved by applying the above result.

3^0 Let $r(z) = 0$ and $Q(x, y) = a(x)$. Then the following equation

$$y'' + F''(y)F'(y^{-1}y'^2 + a(x)y' + c \exp(-2 \int a\,dx)q^{-3}F'(y)^{-1}g(F(y)/q)-$$
$$-(q''/q + a(x)q'/q)F(y)F'(y)^{-1} = 0, \quad (c = \text{const})$$

has the general solution

$$(1.9) \qquad \int (-2 \int g(F/q)d(F/q) + C)^{-1/2}d(F/q) = \int q^{-2} \exp(- \int a\,dx)dx + D$$

$(C, D$ are arbitrary constants, $F = F(y))$.

In particular, the following equations

$$y'' + (k-1)y^{-1}y'^2 + a(x)y' + c \exp(-2 \int a\,dx)y^{1-k}q^{-3}g(y^k/q)-$$
$$-(q''/q + a(x)q'/q)y/k = 0,$$
$$y'' + ky'^2 + a(x)y + c\exp(-2 \int adx)q^{-3}e^{-ky}g(e^{ky}/q) - (q''/q + a(x)q'/q)/k = 0$$
$$y'' - y'^2/y + a(x)y' + c\exp(-2 \int adx)yq^{-3}g(\log y/q) - (q''/q + a(x)q'/q)\log y/y = 0$$

have the general solutions given by (1.9), where $F(y) = y^k$, $F(y) = e^{ky}$, $F(y) = \log y$, respectively. The above equations (first and third) appear in [**12**].

    4^0 Equations

$$y'' + F''(y)F'(y)^{-1}{y^2}' + a(x)y' + \sum_{i=1}^{m} b_i(x)F(y)^{n_i}F'(y)^{-1} = 0,$$

$$y'' + k{y^2}' + a(x)y' + \sum_{i=1}^{m} b_i(x)\exp((n_i - 1)ky)/k = 0,$$

$$y'' + (k-1)y^{-1}{y'}^2 + a(x)y' + \sum_{i=1}^{m} b_i(x)y^{(n_i-1)k+1}/k = 0,$$

where
(1.10)
$$b_i(x) = c_i \exp(-2\int a\,dx)(C_1 + C_2\int\exp(-\int a\,dx)dx + C_3(\int\exp(-\int a\,px)dx)^2)$$

$(C_1, C_2, C_3, c_i, n_i)$ are constants, $i = 1, \ldots, m)$, are integrable by quadratures.

    The above equations are equivalent to

$$Y'' + a(x)Y' + \sum_{i=1}^{m} b_i(X)Y^{n_i} = 0,$$

which is also integrable if b_i have the form (1.10) (see [**5**]).

    $5°$ Let $q(x) = 1$, and $Q(x, y) = a(x)$. Then the following equation

$$y'' + F''(y)F'(y)^{-1}{y'}^2 + a(x)y + c\,\exp(-2\int a\,dx)F'(y)^{-1}g(F(y) - r) -$$
$$-(r'' + a(x)r')F'(y)^{-1} = 0, \quad (c = \text{const})$$

has the general solution

(1.11) $\int(-2\int g(F - r)d(F - r) + C)^{-1/2}d(F - r) = \int\exp(-\int a\,dx)dx + D$

$(C, D$ are arbitrary constants).

    $6°$ Equations

$$y'' + F''(y)F'(y)^{-1}{y'}^2 + a(x)y' + \sum_{i=1}^{m} b_i(x)\exp(n_iF(y))F'(y)^{-1} = 0,$$

$$y'' - {y'}^2/y + a(x)y' + \sum_{i=1}^{m} b_i(x)y^{n_i+1} = 0,$$

where $n_i (i = 1, \ldots, m)$ are constants and b_i are given by

(1.12) $b_i(x) = c_i \exp(-2\int a\,dx - n_i r(x))$ $(c_i, n_i$ are constants $i = 1, \ldots, m)$

$$r(x) = C_1 + C_2\int\exp(-\int a\,dx)dx + C_3(\int\exp(-\int a\,dx)dx)^2$$

$(C_1, C_2, C_3$ are constants), are integrable. This follows from the equation

$$Y'' + a(x)Y' + \sum_{i=1}^{m} b_i(x)\exp(n_i Y) = 0,$$

which is integrable if b_i iz given by (1.12) (see [**5**]).

    2. Let he equation (0.2) be of the form:

(2.1) $\qquad\qquad Y'' + a(x)Y' + b(x)Y + c(x)Y'' = 0 \ \ (n = \text{const}),$

$(a, b, c$ are given functions). Equation of the type (0.1) corresponding to (2.1) is:

(2.2) $\qquad\quad y'' + P(y)y'^2 + a(x)y' + b(x)Q_1(y) + c(x)Q_2(y) = 0,$

where P, Q_1, Q_2, satisfy the following system

(2.3) $\qquad Q_1'(y) = 1 - P(y)Q_1(y), \ \ Q_1(y)Q_2'(y) = (n - P(y)Q_1(y))Q_2(y).$

    Equation (2.2) — (2.3) reduces to the (2.1) by the transformation

(2.4) $\qquad\qquad\qquad Y = F(y) = (Q_2(y)/Q_1(y))^{1/(n-1)}.$

    For the equation (2.1) the following result is known (see e.g. [**7**]):

    Equation (2.1) has the solution $Y = u(x)Z(v(x)/u(x))$, where u and v are linearly independent solutions of the linear equation:

(2.5) $\qquad\qquad\qquad u'' + a(x)/u' + b(x)u = 0.$

with the Wronskian $W = v'u - u'v$; Z is a solution of the following nonlinear equation

(2.6) $\qquad\qquad\qquad d^2 Z/dt^2 + h(t)Z^n = 0,$

where h is determined by

(2.7) $\qquad\qquad\qquad h(v(x)/u(x)) = u(x)^{n+3}c(x)W^{-2}.$

    Using the above we can formulate the following result for the equation (2.2)—(2.3):

    Equation (2.2)—(2.3) has the solution

(2.8) $\qquad\qquad\qquad F(y) = u(x)Z(v(x)/u(x)),$

where u and v are linearly independent solutions of (2.5); F is given by (2.4); Z satisfies the equation (2.6)—(2.7).

    Remarks and *examples*. 1° Using (2.4) we obtain that the equation (2.2)—(2.3) can be represented in the following form:

$$y'' + F''(y)F'(y)^{-1}y'^2 + a(x)y' + b(x)F(y)F'(y)^{-1} + c(x)F(y)^n F'(y)^{-1} = 0.$$

2° Let $h(t) = k(C_1 + C_2 t + C_3 t^2)^{-(n+3)/2}(k, C_1, C_2, C_3, = \text{const})$. Then the equation (2.6) is integrable by quadratures (see e.g. [**7**]) and from (2.7) we find the function c:

$$(2.9) \qquad c(x) = kW^2(C_1 u^2 + C_2 uv + C_3 v^2)^{-(n+3)/2}.$$

In this case the equation (2.2) — (2.3) — (2.11) has the solution

$$(2.10)\quad F(y) = (C_1 u^2 + C_2 uv + C_3 v^2)^{1/2}U(\int (C_1 + C_2 v/u + C_3 v^2/u^2)^{-1}d(v/u)),$$

where $U(s)$ is given by

$$\int (C - (C_1 C_3 - C_2^2/4)U^2 - 2kU^{n+1}/(n+1))^{-1/2}dU = s + D \ (n \neq -1),$$

$$(2.11)\quad \int (C - (C_1 C_3 - C_2^2/4)Y^2 - 2k\log U)^{-1/2}dU = s + D \ (n = -1).$$

(C, D are arbitrary constants, F is given by (2.4)).

3° Furthermore, let

$$(2.12) \qquad c(x) = W^2 u^{-n-3}(\sum_{i=1}^{m}(a_i(v/u)^{b_i})^q \sum_{i,j=1}^{m} A_{ij}(v/u)^{B_{ij}}),$$

where $q = k(1-n) - 2$, $A_{ij} = ka_i a_j b_i(1 - b_i + (1-k)b_j)$, $B_{ij} = b_i + b_j - 2, k, a_i, b_i$ are constants $(i, j = 1, \ldots, m)$.

In this case the equation (2.2)—(2.3)—(2.12) has the solution:

$$(2.13) \qquad F(y) = u(\sum_{i=1}^{m} a_i(v/u)^{b_i})^k$$

where F is determined by (2.4).

4° In particular if $C_1 = 1$, $C_2 = C_3 = 0$, the equation (2.2) — (2.3) — (2.9) becomes

$$(2.14) \qquad y'' + P(y)y'^2 + a(x)y' + b(x)Q_1(y) + k(W^2 u^{-n-3}Q_2(y) = 0$$

(P, Q_1, Q_2 satisfy (2.3)) and has the general solution given by (2.10) — (2.11). with $C_1 = 1$, $C_2 = C_3 = 0$.

In the special case $n = -3$ the equation (2.14) reduces to the well known Herbst's equation (see e.g. [**3**, pp. 62–64], [**4**, pp. 187–188] [**6, 8, 9**]). Also, from the above we obtain the general solution of the Herbst's equation.

If $a(x) = 0$, $b(x) = b =\text{const}$, $c(x) = c =\text{const}$ and $n = 3$ equation (2.14) — (2.3) reduces to the equation (2.5) from [**15**].

5° Let $P(y) = (p-1)/y \ (p \neq 0)$. Then the equation (2.2) becomes

$$(2.15) \qquad y'' + (p-1)y^{-1}y'^2 + a(x)y' + b(x)p^{-1}y = c(x)p^{-1}y^{(n-1)p+1} = 0.$$

In this case $F(y) = y^p$. Some special cases of the equation (2.15) with $c(x)$ given by (2.15) and $k = 2/(n-1)$, i.e. $q = 0$ are treated in [**13, 14, 15**]. These results can be obtained as a particular cases of the above. Also, equation (2.15) is treated in [**17**].

<div align="center">REFERENCES</div>

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