A NOTE ON EXTENSIONS OF BEAR AND P. P.-RINGS

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Bear rings are rings in which the left (right) annihilator of each subset is generated by an idempotent [2]. Closely related to Bear rings are left P. P. -rings; these are the rings in which each principal left ideal is projective, or equivalently, ring in which the left annihilator of each element is generated by an idempotent. In [1] Armendariz showed that if \( R \) is a ring which has no nonzero nilpotent elements then \( R[X] \) is a Bear or P.P.-ring if and only if \( R \) is a Bear or P.P.-ring. In this note we generalize this result. A semigroup \( G \) is called an u.p. semigroup if, when \( A \) and \( B \) are nonempty finite subsets of \( G \), then there always exists at least one \( x \in G \) which has an unique representation in the form \( x = ab \) with \( a \in A \) and \( b \in B \). We prove that if \( R \) is a reduced ring and \( Ga \) u. p. semigroup then the semigroup ring \( RG \) is a Bear or P.P.-ring if and only if \( R \) is a Bear or P.P.-ring.

We will assume throughout that rings have a unit. In a reduced ring left and right annihilators coincide for any subset \( U \) of \( R \), hence we let \( \text{ann}_R(U) = l(U) = \{ a \in R : aU = 0 \} \).

The key lemma is the following characterization of zero divisors in \( RG \) when \( R \) is a reduced ring.

**Lemma 1.** [3, Corollary 3.2] Let \( G \) be an u. p. semigroup and let \( R \) be a reduced ring. Let \( G \) be an u.p. semigroup and let \( p, q \in RG \) such that \( pq = 0 \). Then for any \( g, h \in G \) we have \( p_gq_h = 0 \).

**Corollary 1.** If \( R \) is a reduced ring and \( f \in RG \), \( G \) an u.p. semigroup, such that \( f^2 = f \) then \( f \in R \).

**Proof.** Let \( f = \sum_{i=1}^{n} a_i g_i \). It is easy to show that \( g_i = e \) for at least one \( i \).

Hence we may, without any loss in generality, put \( f = a_1 e + a_2 g_2 + \ldots + a_n g_n \). Now \( f(f - 1) = 0 \). From Lemma 1 we have \( a_1(a_1 - 1) = 0 \) and \( a_i = 0 \) for \( i \geq 2 \). Hence \( f = a_1 = a_1^2 \in R \).

If \( f \in RG \) and \( f = \sum_{i=1}^{n} a_i g_i \) let \( S_f = \{ a_1, a_2, \ldots, a_n \} \).
Corollary 2. Let $R$ be a reduced ring and $U \subseteq RG$. If $T = \bigcup_{f \in U} S_f$ then
\[ \text{ann}_{RG} U = \text{ann}_R(T)G. \]

Proof. This follows easily from Lemma 1.

Theorem 1. Let $R$ be a reduced ring and $G$ an u.p. semigroup. Then $RG$ is a P.P.-ring if and only if $R$ is a P.P.-ring.

Proof. If $RG$ is a P.P.-ring and $a \in R$ then $\text{ann}_R(a) = R \cap \text{ann}_{RG}(a) = R \cap (RG)e$ with $e^2 = e$. By Corollary 1, $e \in R$ and thus $R \cap RG = Re$.

Now assume $R$ is a P.P.-ring. Let $a, b \in R$ with $\text{ann}_R(a) = Re_1$, $\text{ann}_R(b) = Re_2$, where $e_1^2 = e_1, e_2^2 = e_2$. Put $e = e_1e_2$. Because the idempotents of $R$ are central we have $e^2 = e$. We show that $\text{ann}_R \{a, b\} = Re$. If $xa = xb = 0$ then $x = xe_1 = xe_2$ and $xe = xe_1e_2 = x$. Hence $\text{ann}_R \{a, b\} \subseteq Re$. Further, let $t \in Re$, say $t = re_1e_2$. Now $ta = re_1e_2a = re_2e_1a = 0$ and $tb = re_1e_2b = 0$. Hence $Re \subseteq \text{ann}_R \{a, b\}$. Therefore, $Re = \text{ann}_R \{a, b\}$. Thus for any finite subset $U \subseteq R$, $\text{ann}_R(U) = Re$ for some idempotent $e \in R$. If $f \in RG$ then by Corollary 2, $\text{ann}_{RG}(f) = \text{ann}_R(S_f)G = (Re)G = (RG)e$ with $e^2 = e$, as $Sf$ is finite. Thus $RG$ is a P.P.-ring.

Similarly we can establish

Theorem 2. Let $R$ be a reduced ring and $G$ an u.p. semigroup. Then $RG$ is a Bear ring if and only if $R$ is a Bear ring.

Corollary 3 [1, Theorem A] Let $R$ be a reduced ring. Then $R[x]$ is a P.P.-ring if and only if $R$ is a P.P.-ring.

Proof. It follows from the fact that the infinite cyclic semigroup $\langle X \rangle$ is an u.p. semigroup.

Corollary 4 [1, Theorem B]. Let $R$ be a reduced ring. Then $R[x]$ is a Bear ring if and only if $R$ is a Bear ring.

References


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