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GENERALIZATIONS OF SOME THEOREMS OF LOOMIS ON ALMOST PERIODIC FUNCTIONS

G. Crombez

Abstract. We obtain characterizations of almost periodic and weakly almost periodic functions on an arbitrary locally compact Hausdorff group. They include in particular some results of Loomis giving sufficient conditions for a function on an Abelian group to be almost periodic.

1. Introduction. The theorems referred to in the title are the theorems 1, 4 and 5 in [**6**]; they all refer to the question under what conditions a bounded measurable function on a locally compact Abelian group G is almost periodic. The generalizations we obtain are in different directions; first of all, it is no longer necessary that G be Abelian; secondly, we obtain in fact characterizations of the almost periodic functions (and not only implications in one direction) in which the results of Loomis are included; finally, we also get an analogous characterization for weakly almost periodic functions of which there is no analog in the paper of Loomis.

In order to state the results we use the so-called τ_c-and τ_w-topologies (or strict topologies) on $L_\infty(G)$ (see [**1, 2, 8**]) where G is a locally compact Hausdorff group with a fixed left Haar measure on it. For convenience we repeat here their definitions. The space $L_\infty(G)$ may be embedded into the set $B(L_1(G), L_\infty(G))$ of all bounded linear operators from $L_1(G)$ to $L_\infty(G)$ by the operator Φ such that $\Phi(g)(f) = f * g$ ($f \in L_1(G)$, $g \in L_\infty(G)$, $*$ the convolution product). Since $B(L_1(G), L_\infty(G))$ carries naturally the strong and the weak operator toplogy, Φ allows us to consider their induced topologies on $L_\infty(G)$, which we denote by τ_c and τ_w respectively. Clearly $\tau_w \leq \tau_c \leq \| \|\infty$.

In what follows G is an abritrary locally compact Hausdorff group in section 2, and an Abelian one in section 3.

2. Main results. We call a function g in $L_\infty(G)$ right τ_c-almost periodic ($r - \tau_c$-a.p.) iff the set $\{g_a : a \in G\}$ of right translates of g is relatively compact

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with respect to τ_c; we denote the set of these functions by $R-\tau_c-AP$. Analogously, using the τ_w-topology we may define the set $R-\tau_w-AP$. It may be verified that both sets are right invariant linear subspaces of $L_\infty(G)$.

We denote by $AP(G)$ and $W(G)$ the sets of (norm) almost periodic and weakly almost periodic functions in $L_\infty(G)$, respectively. If $\Phi : L_\infty(G) \to B(L_1(G),\ L_\infty(G))$ is the operator defined in the introduction, and if we put $A = \{(\Phi(g))_a : a \in G\}$ where $(\Phi(g))_a(f) = (f * g)_a$ with $f \in L_1(G)$ and $g \in L_\infty(G)$, then an adaption of exercise VI. 9.2 in [**3**] leadds to the following propositional.

Proposition 1. $g \in R-\tau_c-AP \Leftrightarrow f * g \in AP(G),\ \forall f \in L_1(G)$

$$g \in R-\tau_w-AP \Leftrightarrow f * g \in W(G),\ \forall f \in L_1(G).\ \square$$

Due to the fact that $AP(G)$ and $W(G)$ are essential Banach modules over $L_1(G)$, we easily derive from Proposition 1 that $L_1(G) * R-\tau_c-AP = AP(G)$, and $L_1(G) * R-\tau_w-AP = W(G)$. We now state our principal theorems. By $C_{ru}(G)$ we denote the set of all bounded right uniformly continous functions on G[**4, 19, 23**].

Theorem 2. $AP(G) = C_{ru}(G) \cap R-\tau_s-AP$

Proof. That $AP(G)$ is a part of $C_{ru}(G) \cap R-\tau_s-AP$ is immediately clear, since an almost periodic function is necessarily (right) uniformly continuous, and since the τ_c-topology is weaker than the norm topology on $L_\infty(G)$. Conversely, if g belongs to $C_{ru}(G) \cap R-\tau_{rs}-AP$, and $(e_\lambda)_{\lambda\in\Lambda}$ is an approximate identity in $L_1(G)$, then each function $e_\lambda * g$ is almost periodic, by proposition 1. But since g also belongs to $C_{ru}(G)$ we have $\lim\limits_{\lambda} \| e_\lambda * g - g \|_u = 0$ $[5, 32.48]$. Finally, since $AP(G)$ is closed for the uniform norm $\|\ \|_u$ we obtain that g is almost periodic. $\square$

Completely analogously we obtain

Theorem 3. $W(G) = C_{ru}(G) \cap R-\tau_w-AP$.

3. Connection with the results of Loomis. From now on G will always mean a locally compact Ableian group with dual $\widehat{G}$, unless the contrary is explicitly stated. In [**6**] the following definition was introduced.

Definition 1. A bounded measurable function g on G is called almost periodic at the point $\gamma_0 \in \widehat{G}$ iff there exists a function f in $L_1(G)$ such that $f * g$ is $(\|\ \|_\infty-)$ almost periodic and $\hat{f}(\gamma_0) \neq$ ($\hat{f}$ is the Fourier transform of f). Although this definition is rather different from the notion of an $r-\tau_c$-a.p. function, we proved in [**1**. Theorem 2.7] the following result.

Proposition 4. $g \in R-\tau_c-AP$ *iff* g *is almost periodic at each point of* $\widehat{G}$.$\square$

We may remark that for a non Abelian compact group G the notion of $r-\tau_c-a.p.$ function is an agreement with the definition of Loomis. Indeed, for G compact we have that $R-\tau_c-AP = L_\infty(G)$, since for fixed $g \in L_\infty(G)$ the map $a \to g_a$ from G to $(L_\infty(G), \tau_c)$ is continuous; given $g \in L_\infty(G)$ and $\sigma \in \Sigma$, where Σ is the dual object of G [**5**, §**27**, §**28**] there always exists a function f in $L_1(G)$ such that $f*g$ is almost periodic and $\hat{f}(\sigma) \neq 0$; for, any function f in $L_1(G)$ gives rise to an almost periodic function $f*g$, since this last function is continuous; choosing then for f the function $f(x) = X_\sigma(x)$, where X_σ is the character of (a representation in) σ, we obtain $f(\sigma) = I_{H_\sigma}/d_\sigma$ where H_σ is the representation space of a representation corresponding to σ, d_σ is the dimension of H_σ, and I_{H_σ} is the identity operator on H_σ; clearly, then, $\hat{f}(\sigma)$ is different from zero.

We now state the theorems 1,4 and 5 in [**6**] as L_1, L_4, L_5:

L_1: *If g has compact spectrum, then g is almost periodic iff it is almost periodic at each point of G.*

L_4*: If the spectrum of g is compact and residual, then g is almost periodic.*

L_5*: If g is uniformly continous and the spectrum of g is residual, then g is almost periodic.*

We now want to verify that L_1, L_4 and L_5 are contained in our Theorem 2.

In order to see that L_1 is contained in our Theorem 2 it is clearly sufficient in view of Proposition 4, to show that any function g in $L_\infty(G)$ with compact spectrum belongs to $C_{ru}(G)$. The proof of this goes as follows: let I_g be the closed ideal in $L_1(G)$ of function h such that $h*g=0$; ther exists a function f in $L_1(G)$ such that $\hat{f}=1$ on a neighborhood V of the spectrum of g; for any k in $L_1(G)$ we then have that the Fourier transform of $k-k*f$ is zero on V; by [**7**, 7.2.5] this implies that $k-k*f$ belongs to I_g; hence $k*g-k*f*g=0$ for any k in $L_1(G)$, and so $g=f*g$, whence we conclude that g belongs to $C_{ru}(G)$.

Since L follows from L_4 by uniform approximation and since $C_{ru}(G) \cap R-\tau_c-AP$ is uniformly closed, it is sufficient to show that L_4 is contained in our Theorem 2; hence it is sufficient to show that a function g with compact residual spectrum belongs to both the set of right uniformly continuous functions and to the set of functions almost periodic at each point of the dual; but the first part of this has been proved just before, while the second part follows from the proof of Theorem 4 in [**6**].

Remark. Changing in Definition 1 the words almost periodic into weakly almost periodic, we obtain the definition of a function which is weakly almost periodic at a point of the dual group. Then making the changes in Proposition 4 of our paper will give rise to another description of the set $R-\tau_w-AP$, and Theorem 1 in [**6**] will give a new theorem if also there we change almost periodic into weakly almost periodic.

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Seminar of Higher Analysis
State University of Ghent
Galglaan 2
B-9000 Gent (Belgium)

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