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SOME CHARACTERISTICS OF THE PROCESS MEASURE OF THE AMOUNT OF INFORMATION

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Signs and symbols. $a = a_1a_2\dots a_n$ – binary word of length n.

Λ – empty word.

X – the space of all finite words over $\{0, 1\}$. ($\Lambda \in X$ by definition)

$l(a)$ – the length of word a.

$\overline{a} = a_1a_1a_2a_2\dots a_na_n01$ – manner of recording the word a required to record two or more words in the form of one word. For example for the words x, y and z the record is $\overline{x}\,\overline{y}z$. From the word $\overline{x}\,\overline{y}z$ it is possible to decode the words x, y or z by means of general, recursive functions π_1, π_2 and π_3. (We also have $\overline{\Lambda} = 01$.)

$a \subset b$ means $b = aw$, $w \in X$ (aw is a concatenation of words a and w).

$f(x) \preccurlyeq g(x)$ means $(\exists C)(\forall x \in X) f(x) \leq g(x) + C$.

$f(x) \asymp g(x)$ means $f(x) \preccurlyeq g(x)$ and $g(x) \preccurlyeq f(x)$.

The function $F(a_1a_2\dots a_n) = 2^n - 1 + \sum_{i=1}^{n} a_i 2^{n-i}$ gives a one-to-one correspondence of the set X and the set $\{0, 1, 2, \dots\}$. The symbol a will denote both the word and its corresponding number.

Introduction. The partial recursive function $\mathcal{F} : X^{m+1} \to X$ of $m + 1$ arguments is called a process according to argument p if the following applies: for a word p, $\mathcal{F}(p, y_1, \dots, y_m)$ exists and if $q \subset p$, then $\mathcal{F}(q, y, \dots, y_m)$ exists and $\mathcal{F}(q, y_1, \dots, y_m) \subset \mathcal{F}(p, y_1, \dots, y_m)$.

Definition 1. The conditional process complexity of $(x_1, \dots, x_n)$, given $(y_1, \dots, y_m)$, with respect to the processes $\mathcal{F}_1, \dots, \mathcal{F}_n$ is

$$KP_{\mathcal{F}_1,\dots,\mathcal{F}_n}(x_1, \dots x_n / y_1, \dots, y_m) =$$
$$= \min_{p\in X} \{\alpha(p) / \mathcal{F}_1(p, y_1, \dots, y_m) = x_1, \dots, \mathcal{F}_n(p, y_1, \dots, y_m) = x_n\}.$$

The function $\alpha(p)$ is a criterion of complexity and it is usually taken as $\log_2 p$, which in the alphabet $0-1$ is equal to $l(p)+C$.

THEOREM 1. *There is a set of optimal $m+1$ dimensional processes according to argument $p(\mathcal{F}^\circ(p,y_1,\dots,y_m),\dots,\mathcal{F}^\circ_n(p,y_1,\dots,y_m))$ such that for any other set of $m+1$ dimensional processes according to argument $p(\mathcal{G}_1(p,y_1,\dots,y_m),\dots,\mathcal{G}_n(p,y_1,\dots,y_m))$ and for any $(x_1,\dots,x_n)$*

$$KP_{\mathcal{F}^\circ_1,\dots,\mathcal{F}^\circ_n}(x_1,\dots,x_n/y_1,\dots,y_m) \preccurlyeq KP_{\mathcal{G}^\circ_1,\dots,\mathcal{G}^\circ_n}(x_1,\dots,x_n/y_1,\dots,y_m).$$

The proof of Theorem 1. is standard for this theory and similar with the proof in [**2**, p. 91 Theorem 1.2].

From now on, the complexity $KP_{\mathcal{F}^\circ_1,\dots,\mathcal{F}^\circ_n}(x_1,\dots,x_n/y_1,\dots,y_m)$ will be designated with $KP(x_1,\dots,x_n/y_1,\dots,y_m)$. $KP(x_1,\dots,x_n)$ means $KP(x_1,\dots,x_n/\Lambda \dots,\Lambda)$.

We have the following characteristics of the process complexity:

(i) $$KP(x/y) \preccurlyeq KP(x) \preccurlyeq KP(x/y)+2KP(y)$$

where $K(y)$ is the Kolmogorov complexity of the word y. Let $KP(x/y)=l(p)$, that is, $\mathcal{F}^\circ(p,y)=x$. Let us form the function

$$\mathcal{S}=\begin{cases}\mathcal{F}^\circ(\pi_2(z),F^\circ(\pi_1(z))), & \text{if } z \text{ has the form } \overline{a}b\\ \Lambda, & \text{othervise.}\end{cases}$$

$\mathcal{F}^\circ$ is an optimal two-dimensional process, and F° an optimal function for Kolmogorov complexity. Let $K(y)=l(p_y)$. The function $\mathcal{S}$ is a process by construction. For the program $z=\overline{p}_y p$ the results is x. Further more, we have

$$KP(x) \preccurlyeq KP_{\mathcal{G}}(x) \le l(\overline{p}_y)+KP(x/y) \asymp KP(x/y)+2K(y).$$

Remark. The constant 2 may be replaced with $1+\varepsilon$ by a more appropriate coding of the program z.

(ii) $$KP(x/y) \preccurlyeq K(x/y)+2\log_2 K(x/y)$$

Let us form a process

$$\mathcal{J}^2(z,y)=\begin{cases}F^\circ(A(z),y), & \text{if } z \text{ has the form} \overline{a}b \text{ and } l(b)\ge a\\ \Lambda, & \text{otherwise}\end{cases}$$

where $A(\overline{l(p)}pq)=p$ is general recursive $(p,q\in X)$. For $F^\circ(p_x,y)=x$ and $z=\overline{l(p_x)}px$ we have

$$KP(x/y) \preccurlyeq KP_{\mathcal{J}}(x/y) \le l(z)=l(\overline{(p_x)})+K(x/y) \asymp K(x/y)+2l(K(x/y)).$$

(iii) If $\mathcal{F}(x)$ is a process, then $KP(\mathcal{F}(x)) \preccurlyeq KP(x)$.

If for $\mathcal{F}(x)$ there exists an inverse function that is also a process, then $KP(\mathcal{F}(x)) \asymp KP(x)$.

(iv) $$KP(x/y) \succcurlyeq KP(x/y,z) \tag{1.1}$$

$$\begin{aligned}KP(x/y)=\min\{l(p)/\mathcal{F}^\circ(p,y)=x\}&=\min\{l(p)/\mathcal{G}(p,y,z)=x\}\succcurlyeq\\ &\succcurlyeq \min\{l(p)/\mathcal{F}^\circ(p,y,z)=x\}=KP(x/y,z).\end{aligned}$$

The function $\mathcal{G}(p,y,z) = \mathcal{F}^\circ(p,y)$ has z as a fictive argument.

(v) For every partial recursive function F we have

$$KP(y/x, F(x)) \asymp KP(y/x)$$
$$KP(y/x, F(x)) = \min\{l(p)/\mathcal{F}^\circ(p,x,F(x)) = y\} =$$
$$= \min\{l(p)\mathcal{G}(p,x) = y\} \succcurlyeq KP(y/x).$$

(vi) If F is an invertible partial recursive function, then

$$KP(x/F(x)) \asymp KP(F(x)/x) \asymp 0 \tag{1.2}$$
$$KP(x/F(x)) = \min\{l(p)/\mathcal{F}^\circ(p,F(x)) = x\} \preccurlyeq \min\{l(p)/\mathcal{G}(p,F(x)) = x\} \asymp 0,$$

where $\mathcal{G}(p,F(x)) = F^{-1}(F(x))$, which is trivially a process according to p.

$$KP(F(x)/x) = \min\{l(p)/\mathcal{F}^\circ(p,x) = F(x)\} \preccurlyeq \min\{l(p)/\mathcal{G}(p,x) = F(x)\} \asymp 0,$$

where $\mathcal{G}(p,x) = F(x)$, which is also a process according to p.

Measure of the amount of information. The process complexity of a word x is very suitable for defining the concept of randomness. Namely, (Schnornr in [**4**] shows that to a Martin-Löf random binary sequences ω applies $KP(\omega^n) \asymp n$, where ω^n, is a fragment of the sequences ω of length n. On the other hand, the complexity is also suitable for defining the measure of information. Kolmogorov defines in [**1**]) the measure of information carried by a word y about word x as

$$I(y:x) = K(x) - K(x/y) \tag{2.1}$$

Levin ([**5**]) also defines the measure of information as $IP(y:x) = \overline{KP}(x) - \overline{KP}(x/y)$, where $\overline{KP}_A(x) = \min\{l(p)/A(p) = x\}$ and $A(p)$ is a function such if $A(p) = x$, then $A(pq) = x$. (Those are the so-called prefix algorithms.)

Definition 2. The quantity

$$J(y_1,\ldots,y_m : x_1,\ldots,x_n/z_1,\ldots,z_k) = KP(x_1,\ldots,x_n/z_1,\ldots,z_k) -$$
$$-KP(x_1,\ldots,x_n/y_1,\ldots,y_m,z_1,\ldots,z_k)$$

is termed the process measure of the amount of information that $(y_1,\ldots,y_m)$ carries on $(x_1,\ldots,x_n)$ if $(z_1,\ldots,z_k)$ is known. We have the following characteristics of measure J:

(i) $$J(y:x) \succcurlyeq 0 \tag{2.2}$$

The property (2.2) follows from the relation (1.1).

(ii) $$J(x:x) \asymp KP(x) \tag{2.3}$$

The relation (2.3) is a direct consequence of (1.15). It can be also shown that $J(p_x : x) \asymp KP(x)$, where p_x is such that $\mathcal{F}^\circ(p_x) = x$.

(iii) $$J(x,y:z) = J(x:z) + J(y:z/x) \tag{2.4}$$

The proof results directly from the definition of the measure J.

(iv) The process measure of information may be compared with measure I, introduced by (2.1)

$$I(y:x) - 2\log_2 K(x/y) \preccurlyeq J(y:x) \preccurlyeq I(y:x) + 2\log_2 K(x)$$
$$J(y:x) = KP(x) - KP(x/y) \preccurlyeq K(x) + 2\log_2 K(x) - K(x/y) =$$
$$I(y:x) + 2\log_2 K(x).$$

(v) If F is partial recursive and invertible function, $J(F(x):x) \asymp KP(x)$, $J(x:F(x)) \asymp (F(x))$, $J(F(x):y) \asymp J(x:y)$.

(vi) It is known that the algorithm measure of the amount of information is not commutative ([**2**], [**3**]), that is, it can be shown only as $|J(y:x) - I(x:y)| \preccurlyeq 12 \cdot I(K(x,y))$. Since $|J(y:x) - I(y:x)| \preccurlyeq (1+\varepsilon)l(K(x))$, for the process measure J we have
$$|J(y:x) - J(x:y)| \preccurlyeq (14 + 2\varepsilon)l(K(x,y)).$$

(vii) For every word x we have $J(l(x):x) \preccurlyeq 2 \cdot K(l(x))$.

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