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GRAPH EQUATIONS FOR LINE GRAPHS AND n-th DISTANCE GRAPHS

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0. Introduction: In this paper we will investigate the graph equation

$$L(G) = D_n(G), \tag{1}$$

$L(G)$ being a line graph of G, while $D_n(G)$ is the n-th distance graph of G, i.e., $D_n(G)$ is a graph having the same vertex set as G with two vertices u and v being adjacent in $D_n(G)$ if the distance $d_G(u, v)$ between u and v in G equals n. Note that there is a similar operation in literature, namely, the n-th path graph (see [**1**] for a definition).

If $n = 1$, equation (1) becomes a classical one, i.e. $L(G) = G$, which was solved in [**2**]. The case $n = 2$ is in fact, identical with the equation $L(G) = Pn(G)$, where n denotes the n-th path graph (3). Accordingly, we will assume throughout this paper that $n > 2$.

All usual terminology in this paper is taken from (4). Here we quote some unusual terminology. We say that graph is even (odd), if all its cycles have even (odd) lengths. $G\langle v_1, \dots, v_m\rangle$ denotes an induced subgraph of G obtained by taking the vertices $v_1, \dots, v_m$ as its vertex set. If H is an induced subgraph of G, we write $H \subseteq G$. In general, if H is any subgraph of G we write $H \leq G$. H is a distance-preserving subgraph of G, if H, as a subgraph of G, satisfies $d_H(u, v) = d_G(u, v)$, for any pair u, v of vertices from H. The graph $C_m(k_1, \dots, k_m)$ is obtained as follows: we take a cycle C_m of length m (the vertices of the cycle are labeled from 1 to m) and append to the i-th vertex a path of length k_i.

We also make some general remarks concerning the equations of the type

$$F(G) = G, \tag{2}$$

where F is assumed to be the graph valued function which is additive with respect to union (of graphs) and also preserves the connectedness of components. Taking

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$G = \bigcup G_i$ (G_i is connected) as a solution to (2), we easily get that for some permutation $\pi F(G_i) = G_{\pi(i)}$ holds for each i. Recalling the cycle structure of π, we conclude that the set of components can be partitioned in such a way that each part contains just the components which can be obtained one from each other using the iterations of F. In other words, after the appropriate relabeling of components, it follows that any part takes the form:

$$G_1, \dots, G_i, \dots, G_p, \tag{3}$$

where for each $i = 1, \dots, p$ $G_i = F^{i-1}(G_1)$, while $F^p(G_1) = G_1$. The number of parts clearly equals the number of cycles of π, while the number of graphs in each part coincides with the length of the corresponding cycle. Assuming the labeling from (3) we can say that

$$G^* = \bigcup_{i=1}^{p} G_i \tag{4}$$

is also a solution to (2). Any solution of this type (which cannot be splitted into some other ones) will be referred to as a fundamental solution. Namely, any other solution can be represented as the union of the fundamental ones. So, in order to solve the equations of the type (2), we only need to find its fundamental solutions. Any of the graphs $G_1, \dots, G_p$ appearing in (3) may be referred to as a generator of the corresponding fundamental solution. The integer p is assumed to be the period of any graph $G_1, \dots, G_p$, since it is the smallest integer such that for each $i = 1, \dots, p$, $F^p(G_i) = G_i$ holds. The above conclusions can be summarized in the following proposition.

Proposition 1. *Under the assumptions above, the equation (2) is equivalent to the following family of equations*

$$F^p(G) = G, \tag{5}$$

p being a natural number, while G is assumed to be connected.

Clearly, the same argument applies to the equations $f(G) = g(G)$, if f or g is invertible at least on the solution set of the equation.

1. Main considerations: From now on we shall assume that G denotes a possible solution to (1). If so, suppose that G is a fundamental solution as well. To find it, it is enough to find any of its generators, and their common period if possible. By (m, n) we denote the greatest common divisor of m and n.

Lemma 1. *The cycle C_k is a component of G if and only if $k > 2n$ and $(k, n) = 1$. Moreover, C_k is a fundamental solution as well.*

Proof. Let $d = (k, n)$. Then

$$D_n(C_k) = \begin{cases} kK_1 & k < 2n \\ nK_2 & k = 2n \\ dC_{k/d} & k > 2n \end{cases}$$

Now the proof of the Lemma immediately follows. □

Because of this lemma, from now on, we will ignore the cycles as components of G.

LEMMA 2. *Every component of G which is not a cycle is unicyclic and odd.*

Proof. Clearly, the number of points $p(G)$ equals the number of edges $g(G)$. Thus, to prove the unicyclic part, it is sufficient to show that G has no acyclic components. Recalling that for any graph H we have $\kappa(L(H)) \leq \kappa(H) \leq \kappa(D_n(H))$, where κ denotes the number of components of the corresponding graph, we obtain

$$\kappa(L(G)) = \kappa(G) = D_n(G)). \tag{6}$$

Clearly, no component of G is trivial. If any, say G_i, is a tree, then for n even, $D_n(G_i)$ is disconnected, which contradicts (6); for n odd, $D_n(G_i)$ is bipartite (and a line graph as well), implying that $D_n(G_i)$ is a path or a cycle which in turn implies the same for some components of G. This is again impossible by (6); or the assumptions.

Suppose now some component of G is unicyclic but even. Discussing the parity of $n4$, we obtain contradictions analogous to those above. □

LEMMA 3. *The maximal vertex degree of G is less than* 4.

Proof. Suppose the contrary and let v be a vertex of G such that $\deg v \geq 4$. Then $K_4 \subseteq L(G)$ and also $K_4 \subseteq D_n(G)$. Hence we can find in G four vertices $v_1, \dots, v_4$ at a mutual distance which is exactly n. Consequently, there is a subgraph of G, say H, such that

(a) H is distance preserving,

(b) H contains the verticesv_1, v_2, v_3,

(c) H is critical with respect to (a) and (b).

By a slight effort it follows that H must be one of the graphs H_1 or H_2 (see Fig. 1), depending on whether paths between v_1, v_2, v_3 have any vertex in common.

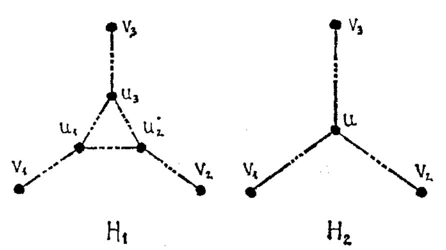

Fig. 1.

Case 1: H_1 appears in G. We now easily get

$$d(v_i, u_i) = d(u_j, u_k) + (n - g)/2 \quad (i \neq j \neq k \neq i), \tag{7}$$

where $g(= d(u_1, u_2) + d(u_2, u_3) + d(u_3, u_1))$ is, in fact, the girth of the component that contains H_1. Since G is odd, by Lemma 2, the same follows for n. Hence H_1 may appear only for n odd. Next, consider v_4 as well. Of course, it cannot be in H_1. Let u_4 be a vertex of H_1 such that $d(v_4, u_4)$ is minimal. Clearly, $d(v_4, v_i) = d(v_4, u_4) + d(u_4, v_i) = n$, implying that all $d(u_4, v_i)$ are of the same parity. Since $d(v_i, u_4) + d(u_4, v_j) = n$ for some i, j we get a contradiction.

Case 2: H_2 appears in G. Now, we have

$$d(v_i, u) = n/2 \tag{8}$$

and this implies that n is even. Hence H_2 may appear in G only if n is even. Next, as above, consider v_4 and note that it cannot be in H_2. Let u_x be a vertex of H_2 such that $d(v_4, u_x)$ is minimal. Now we easily get $u_x = u$. Thus for all $i = 1, \ldots, 4$, we have $d(v_t, u) = n/2$. Since u cannot be isolated in $D_n(G)$ by (6), there is a vertex of G at distance n from u, and consequently, a vertex, say w, such that $d(u, w) = n/2 + 1$. Without loss in generality, let w be adjacent to v_4; also let w_1, w_2, w_3 be the neighbors of v_1, v_2, v_3 such that $d(u, w_i) = n/2 - 1$ (see Fig. 2).

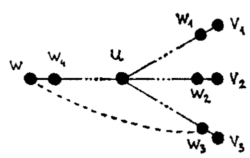

Fig. 2.

Of course, $d(w, w_i) \le n$. Assume first, $d(w, w_i) = n$ for all i. But then $D_n(G) \langle w, w_1, w_2, w_3 \rangle = K_{1,3}$ (note that $d(w_i, w_j) < n$). Thus, say $d(w, w_3) = k$, where $k < n$. On the other hand, $d(v_3, v_4) \le d(v_3, w_3) + d(w_3, w) + d(w, w_4) = k+2$, and hence $k \ge n - 2$. Moreover, if v_3 or v_4 belongs to the shortest path between w and w_3, then $d(v_3, v_4) < n$. Now if $k = n - 2$, we can find in G two different paths between v_3 and v_4, each being of length n. But this implies that G contains an even cycle, which contradicts Lemma 2. So assume $k = n - 1$. But then, since $d(w, w_i) = n - 1$ must hold for all i, or just for one i, we have $D(G) \langle w, v_1, v_2, v_3, v_4 \rangle = K_5 - x$ or $D_n(G) \langle w, v_1, v_2, v_3 \rangle = K_{1,3}$, an obvious contradiction. $\square$

The next lemma easily follows from the proof of the previous one.

LEMMA 4. *Any triangle of $D_n(G)$ originates from some distance-preserving subgraph of G which is (depending on the parity of n) equal either to H_1 or H_2 (see Fig. 1).*

In the next lemma we assume that $G_u \cdot H_v$ is a dot product of rooted graphs obtained by identifying their roots.

LEMMA 5. *No component of G is equal to the graphs*

$$C_m(k, 0, \ldots, 0) \quad (k > 1), \quad \text{or} \tag{9}$$

$$C_3 \cdot T_u \quad (T_u \text{ is a rooted tree}). \tag{10}$$

Proof. The graphs (9) are eliminated by direct inspection. To eliminate the graphs (10) observe first that $D_n(C_3 \cdot T_u) = D_n(C_3 \cdot T_u - x)$, where x is an edge of C nonincident with u (note, u is identified with some vertex of C_3). The rest of the proof is as for Lemma 2. □

In order to prove the next lemma assume for a moment that any component of G is different from the graph $C_m\,(k, 0, \ldots, 0)$ where $k \leq 1$. Namely, $k = 0$ is already assumed, while for $k = 1$, the corresponding graph, as will be pointed later on, is a generator of some fundamental solution with a period equal one. Hence, for any component G_i of G, $D_n(G_i)$, since being equal to $L(G_j)$ for some j, has at least two triangles.

LEMMA 6. *G has no triangles.*

Proof. Suppose G_i is a component containing a triangle.

Case 1: n is odd. By Lemma 4, the graph H_1 of Fig. 1 appears now in G_i as a distance-preserving subgraph. Moreover, at least two copies of H_1 must appear in G_i each of them having at least a triangle in common. If there are just two copies of H_1 in G_i, then $D_n(G_i)$ contains only two triangles which have an edge in common, and this contradicts (10). If more than two copies of H_1 appear in G_i, then $D_n(G_i)$ contains either $K_{1,3}$ or C_4 as an induced subgraph, an obvious contradiction.

Case 2: n is even. By Lemma 4 again, graph H_2 of Fig. 1 is now a distance-preserving subgraph of G_i. Let H_v be a copy H_2 with a central vertex v. Choose t to be a vertex of the triangle of G_i, so that $d(v,t)$ is minimal. If $n/2 \leq d(v,t) \leq n-2$ (or $d(v,t) \geq 3n/2$), then C_4 (respectively $K_{1,3}$) appears in $D_n(G_i)$, a contradiction. Note also that at most two subgraphs H'_v and H''_v may exist in G_i; otherwise C_4 or $K_{1,3}$ appears in $D_n(G_i)$ again. Observe now two subgraphs H_v and H_w of G_i $(v \neq w)$ both equal to H_2. Then H_{vw} (see Fig. 3) appears in G as a distance-preserving subgraph.

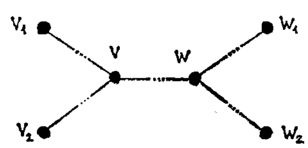

Fig. 3.

Now $d(v,w) \leq n$, since otherwise $K_{1,3} \subseteq D_n(G_i)$. If $1 \leq d(v,w) \leq n-2$, we get contradictions as follows. If $d(v,w) = 1$, then $2(K_4 - x) \subseteq D_n(G_i)$. The latter implies that some component of G has at least two triangles. On the other hand, if $2 \leq d(v,w) \leq n-2$, then $C_4 \subseteq D_n(G_i)$. So assume $d(v,w) = n-1$ or n. Also suppose s is a vertex of H_{vw} with degree (in G_i) equal to 3 (more than 3 is impossible by Lemma 3). If s belongs to $v - v_i$ or $w - w_i$ $(i = 1, 2)$ paths of H_{vw}, (note that $s \neq v, w$), then $K_{1,3} \subseteq D_n(G_i)$. Otherwise, if s belongs to $v - w$

paths of H_{vw}, then $C_4 \subseteq D_n(G_i)$ holds always except for the following possibility: $d(v,w) = n-1$ and there exist two vertices s, t on the $v-w$ path of H_{vw} such that $d(v,s) = d(t,w) = n/2-1$, $d(s,t) = 1$ and the edge st belongs to the triangle of G_i. But then $2(K_4 - x) \subseteq D_n(G_i)$, a contradiction as earlier. Thus $D_n(G_i)$ may have at mast two triangles, which have an edge in common. The latter contradicts (10). □

LEMMA 7. *Any vertex of G having degree equal to 3 belongs to a cycle.*

Proof. Suppose the contrary and let G_i be a component of G that contradicts the Lemma. Among the vertices of degree 3 not belonging to C (C is a unique cycle of G_i) choose v so that $d(v,C)$ is as large as possible. Next, let v_1, v_2, v_3 be the neighbors of v among which v is the closest to C. Clearly, v_1, v_2, v_3 are mutually nonadjacent in $D_n(G_i)$ and belong to the same component of $D_n(G)$ (note, $D_n(G_i)$ must be connected). Hence, there must exist in $D_n(G_i)$ a vertex, say u_x, adjscent to at least one vertex among v_1, v_2, v_3. If the position of u_x is abserved in G_i, it follows that u_x must be adjacent to precisely two vertices among v_1, v_2, v_3. If u_x is adjacent in $D_n(G_i)$ to v_3 and v_1 (or v_2), then, due to the maximality in the choice of v, u_x has no more neighbors in $D_n(G_i)$. Next, if u_x is adjacent in $D_n(G_i)$ to v_1 and v_2, then u_x, again, has no more neighbors; otherwise if w is neighbor of u_x, then, to avoid the appearing of $K_{1,3}$ or $K_4 - x$ in $D_n(G_i)$ w must be adjacent, besides u_x, either to v_1 or v_2, which is impossible. It is also forbidden that two vertices u_x and u_y are adjacent to the same pair of vertices among v_1, v_2, v_3; C_4 or $K_4 - x$ appears again in $D_n(G_i)$. Thus, $D_n(G_i)$ contains C_6 as a subgraph, which is clearly a contradiction. □

We now focus our attention on the girth of all components which are not cycles: By C we denote the unique cycle of the component under consideration. If v and C are in the same component, then v_c denotes the vertex of C which is unique of course) for which $d(v,C)$ is minimal: In the following lemmas we will discuss the girth of the components of G.

LEMMA 8. *No component of G has a girth less than $2n-3$.*

Proof. It is sufficient to show that $D_n(G_i)$ is disconnected whenever for some component G_i its girth $g(G_i)$ is less than $2n-3$. To end this, we first split the vertex set of G into two disjoint classes $V_1 = \{v | 1 \leq d(v,C) \leq n-1-h\}$ and $V_2 = \{v | v \in C \text{ or } d(v,C) \geq n-h\}$, where $h = [g(G_i)/2]$. Turning to $D_n(G_i)$, it must contain an edge, say x, such that $x = v_1v_2$ and $v_1 \in V_1$, $v_2 \in V_2$. If so, $d(v_2,C) \geq n-h$ and also $(v_1)_C \neq (v_2)_C$: otherwise, $\deg v_2$ equals 1 in $D_n(G_i)$ and this fact deos not ensure the connectedness of $D_n(G_i)$. So we can find on C two vertices u_1 and u_2 such that $d(v_2,u_1) = d(v_2,u_2) = n$, which together with v_1 and v_2 induce $K1,3$ in $D_n(G_i)$, providing a contradiction. □

In the following series of lemmas we investigate the components of G having girth $2n-1$.

LEMMA 9. *Let G_i be a component of G with $g(G_i) = 2n-1$. If u belongs to C and* $\deg u = 3$, *there exists a vertex v of G such that $d(u,v) = n$ and $d(u,v_c) = [(n-1)/2]$ or $n-1$.*

Proof. Clearly, u cannot be isolated in $D_n(G_i)$. Thus, for some vertex v, we have $d(u,v) = n$. But then, in order to avoid $K_{1,3}$ in $D_n(G_i)$, we must have $d(u, v_c) = [(n-1)/2]$ or $n-1$. □

In the next few lemmas, when there is no indication to the contrary, we assume that n is odd.

LEMMA 10: *The graphs*[1] *of Fig. 4 cannot be the induced subgraphs of G.*

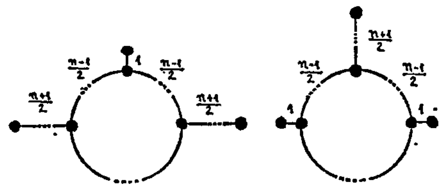

Fig. 4.

Proof. If the graphs above really appear in G, then C_6 appears in $D_n(G)$ as an induced subgraph. The latter contradicts Lemma 2. □

LEMMA 11. *Let G_i be a component of G with $g(G_i) = 2n-1$. Then there is a pair of vertices in G_i not lying on C such that $d(u,v) = n$. In addition $d(u_c, v_c) = [(n-1)/2]$ holds.*

Proof. We first note that any pair of vertices originating from C is nonadjacent in $D_n(G_i)$. Since $D_n(G_1)$ cannot be bipartite, there must exist a pair of vertices u and v, which satisfy the Lemma. If $d(u_c, v_c) \neq (n-1)/2$, then $K_{1,3} \subseteq D_n(G_i)$, a contradiction. □

LEMMA 12. *If G is a component of G with $g(G) = 2n-1$, then $d(v, C) < n$ for any vertex t of G.*

Proof. Suppose the contrary and consider a vertex u such that $d(u, C) \geq n$. Then $d(u, C) = n$ since otherwise $K_{1,3} \subseteq D_n(G_i)$. Let u, v_1, v_2, v_3, v_4 be the vertices of G_i as shown in Fig. 5a.

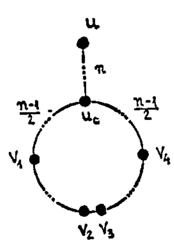

Fig. 5a

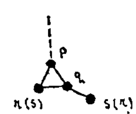

Fig 5b.

[1]The labellings in Fig. 4 stand for the distances between corresponding vertices.

By Lemma 11, all vertices of C except u_c and possibly v_1, v_2, v_3, v_4 are of degree 2. Next, by Lemma 10, $\deg v_1$ or $\deg v_4$ is equal to 2, while, by Lemma 11, either v_1 or v_4 is degree 3. If for some vertex w, $d(w,C) > 0$ and w_c is equal to v_2 or v_3, then, by Lemma 10 or 11, $d(w,C) < (n+1)/2$. Also, if $n > 3$, then $\deg v_2 = \deg v_3 = 3$ cannot hold: otherwise $K_{1,3} \subseteq D_n(G_i)$. If $n = 3$, it easily follows that $d(u,C)$ cannot be equal to n. So, assume $\deg v_1 = 3$, $\deg v_4 = 2$, while $\deg v_2 = 2$ (or 3) and $\deg v_3 = 3$ (or 2). Now, let p, g, r, s be the vertices chosen such that: $d(p,C) = 1$, $p_c = v_1$; $d(q,C) = (n-1)/2$, $q_c = u_c$; $d(r,C) = d(s,C) = 0$, $d(q,r) = d(q,s) = n$. Clearly, p, q, r, s induce in $D_n(G_i)$ a graph equal to $K_3 \cdot K_2$ (see Fig. 5b) and also r and s have no more neighbors in $D_n(G_i)$, as follows from Lemmas 10 and 11. The detail from Fig. 5b cannot appear in a line graph of any component of G under the restriction registered thus far. □

LEMMA 13. *All components of G having girth $2n-1$ have exactly $4n-2$ vertices.*

Proof. Assume that $G_1, \dots, G_p$ is a sequence of components for which

$$D_n(G_i) = L(G_{i+1}) \quad (i = 1, \dots, p; G_{p+1} = G_1) \tag{11}$$

hold. Now it is easy to get that, if $g(G_i) = 2n-1$ for some i, the same holds for each $i = 1, \dots, p$. Next, for each component G_i, let a_i, b_i and c_i denote respectively the number of vertices on the cycle, the number of vertices outside the cycle and the number of vertices of degree 3 (or 1 as well). Also let d_i denote the number of triangles in $D_n(G_i)$. By a simple argument, we have:

$$q(D_n(G_i) = 2a_i + d_i, \quad q(L(G_{i+1})) = a_{i+1} + b_{i+1} + c_{i+1}. \tag{12}$$

From (11) and (12), since $a_i = a_{i+1}$ $(= 2n-1)$ and $c_{i+1} = d)i$ (= the number of triangles on each component side of (11)), we get $a_i = b_{i+1}$. Thus, $a_i + b_i = 4n-2$ for each $i = 1, \dots, p$. □

The next lemma is a direct consequence of Lemma 11.

LEMMA 14. *Let G_i be a component of G with $g(G_i) = 2n-1$. If u is a vertex of G_i such that $d(u,C) = l > 0$ and v is a vertex of C satisfying $n-l-1 \leq d(v,u_c) \leq n-2$, then $\deg v = 2$, except possibly when $d(v,u_c) = (n-1)/2$ (or $n/2-1$ if n is even).*

LEMMA 15. *Let G_i be a component of G with $g(G_i) = 2n-1$. Suppose u and v are the vertices of G_i such thus $d(u,v) = n$, provided none of them is on C. Also, suppose w is a vertex of G_i not on C such that w_c lies between u_c and v_c on the shorter part of C. Then $d(w,C) \leq (n-3)/2$.*

Proof. If $d(w,u)$ or $d(w,v)$ is greater or equal to n, then $K_{1,3} \subseteq D_n(G_i)$. Therefore we have

$$d(u,u_c) + d(u_c,w_c) + d(w_c,w) \leq n-1, \quad d(v,v_c) + d(v_c,w_c) + d(w_c,w) \leq n-1 \tag{13}$$

which implies $d(w,w_c) \leq (n-2)/2$. □

LEMMA 16. *If G_j is a component of G with $g(G_j) = 2n - 1$, then whenever $n > 3$, the vertices of degree 3 in G_j crre nonadjacent.*

Proof. Suppose the contrary. Since $L(G_j) = D_n(G_i)$ for some i, we have $K_3 \cdot K_3 \subseteq D_n(G_i)$. The latter implies that in G_i there exists a vertex, say v, and four other vertices at distance n from v. Using the same arguments as with Lemma 13 we get $g(G_i) = 2n-1$. If v is on the cycle C, then the four vertices mentioned are out of it. As also required, suppose that two of them, say u and w, are at distance n. If so, paths of length n between vertices v, u and v, w must be disjoint. Then we easily get $d(u_c, w_c) = (n+1)/2$, a contradiction by Lemma 11. So, by using the foregoing lemmas, it follows that G_i contains as an induced subgraph the graph of Fig. 6. In addition we bave: $d(v,u) = d(v,w) = n$, $d(v_c,u_c) = d(v,w_c) = (n-1)/2$, $d(v_c,t_1) = d(v_c,t_2) = n-1$. Now, by Lemma 14, there are no vertices of degree 3 between t_1 and t_3, t_2 and t_4 (shorter parts of C are assumed). According to the same lemma some vertices closer to u_c (or w_c) depending on the length $d(w,w_c)$ ($d(u,u_c)$) are of degree 2. Next, assume there is a vertex x between u_c and t_5 (or w_c and t_6) such that $\deg x = 3$. By Lemma 9, there must exist a vertex y such that $d(x,y) = n$ and $d(u,y) = (n-1)/2$ or $n-1$. If $d(x,y_c) = (n-1)/2$, then y_c falls between v_c and w_c. Since $d(y,y_c) = (n-1)/2$, this contradicts Lemma 15. If $d(x,y_c) = n-1$, then $\deg y_c = 2$, as already observed. Thus, besides u_c, v_c, w_c only $t_1, \dots, t_6$ could have degrees equal to 3.

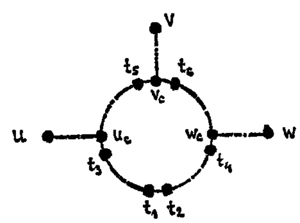

Fig. 6.

For convenience, let $l(x) = \max_{y \in Y_x} d(y,C)$, where $Y_x = \{y | y_c = x\}$ (clearly, x is a vertex of C).

We now make the following observations:

(14) $l(v) \leq (n-1)/2$(see Lemma 10);

(15) $l(t_1)$ or $l(t_5) = 0$, $l(t_2)$ or $l(t_6) = 0$, $l(t_3)$ or $l(t_4) = 0$
(see Lemma 11);

(16) $l(t_3) + l(v_c) + (t_4) = (n-1)/2$
(see (14) if $l(t_3) = l(t_1)$, or (15) and Lemma 11 otherwise).

We now discuss conditions which ensure that t_5 and t_6 are not isolated in $D_n(G_i)$, i.e., we look for the vertices x and y such that $d(t_5,x)$ and $d(t_6,y)$ are equal to n.

Case 1: $l(u_c) \geq (n+3)/2$. Now, $l(w_c) \leq (n-1)/2$ and $l(t_1) = 0$ by Lemma 10; $l(t_2) = l(t_5) = l(t_c) = 0$ by Lemma 11; $l(t_3) + l(v_c) + l(t_4) \leq (n-1)/2$ is just (16). Thus, since $l(u_c) \leq n-1$ (see Lemma 12) we get

$$\sum_{x \in C} l(x) \leq 2n-2, \tag{17}$$

which contradicts Lemma 13.

Case 2: $l(u_c), l(w_c) = (n-1)/2$. Now, $l(t_1) = l(t_2) = l(t_5) = l(t_6) = 0$, by Lemma 11. Observing (16) as well, we get contradiction by (17).

Case 3: $l(t_1), l(t_2) \geq 1$. Now, by (15), $l(t_5) = l(t_6) = 0$; $l(t_1) + l(w_c) \leq (n-1)/2$, $l(t_2) + l(u_c) \leq (n-3)/2$ by Lemma 11. If (16) is observed again, we get the same contradiction as in the previous case.

Case 4: $l(u_c) \geq (n-1)/2$, $l(t_2) \geq 1$. This possibility by itself contradicts Lemma 11.

So t_5 or t_6 is isolated in $D_n(G_i)$, a contradiction. □

LEMMA 17. *If n is odd and $n > 3$, no component of G has the girth equal to $2n-1$.*

Proof. Assume G_i is a component of G for which $g(G_i) = 2n-1$. Since $D_n(G_i)$ cannot be bipartite there are two vertices in G_i, say u and v, such that $d(u,v) = n$, and m addition, neither u nor v belongs to C. Now, G_i contains as an induced subgraph a graph exactly equal to the graph of Fig. 6 with one slight modification; namely, all vertices x for which $x_c = w_c$ may be ignored; the rest is the same. As in the previous lemma, we first conclude that there are no vertices of degree 3 between t_2 and t_4 (the shorter part of C is assumed). Now suppose x is a vertex of degree 3 lying on the shorter part of C between u_c and v_c. Since x cannot be isolated in $D_n(G_i)$ there is a vertex y in G_i such that $d(x,y) = n$. By Lemma 9, $d(x, y_c) = (n-1)/2$ or $n-1$. According to Lemmas 9, 11 and 16, any position of y_c gives a contradiction. Thus, all vertices between u_c and v_c (on the shorter part of C) are of degree 2. So we have to examine whether any vertex of degree 3 can exist between t_1 and u_c, or w_c and v_c (in both cases shorter parts of C are assumed). Considering Lemmas 9 and 16 as well, the only possibility for the existence of such vertices is that they appear in pairs so that their mutual distance is $n-1$. Hence, one is between t_1 and t_3, while the other is between w_c and v_c. Denote these vertices by x and y. Using Lemma 11, we have:

$$l(x) + d(x,u_c) + d(u_c,v_c) + l(v_c) \leq n-1, \;\; l(y) + d(y,v_c) + d(v_c,u_c) + l(u_c) \leq n-1.$$

By adding these relations we get an obvious contradiction. Thus there are no vertices of degree 3 in the corresponding parts of C.

Till now, we have proven that besides u_c and v_c, only w_c, t_1, t_2, t_4 possibly have degrees equal to 3. The rest of the proof runs in the same way as the corresponding part of the proof of the preceding lemma. □

The next lemma stems from the lemma above as a direct consequence.

LEMMA 18. *If $n = 3$ and if G is a component of G with $g(G_i) = 2n-1$ $(= 5)$, then $G_i = C_5(1,1,1,1,1)$; G_i, as a generator of some fundamental solution, has a period equal to 1.*

We conclude consideration of the components of girth $2n-1$ by letting n be even.

LEMMA 19. *If n is even, G has no components of girth $2n-1$.*

Proof. Suppose the contrary and let G_i be a component having the minimal number of vertices of degree 3. Since $D_n(G_i)$ must have at least one triangle, using Lemma 4, we get that the graph of Fig. 7 appears now in G_i as an induced subgraph. Also we must have $d(v, v_c) = n/2$, $d(v, x) = n$, $d(v_c, u_c) = d(v_c, w_c) = n/2 - 1$ and $d(v_c, t_i) = d(v_c, t_2)n - 1$.

Using Lemma 14 (it holds for n even as well), it follows that all vertices between t_1 and u_c, t_2 and w_c (in both cases shorter parts of C are assumed) have degrees equal to 2. By Lemma 9, there exists a vertex, say y, such that $d(x_c, y) = n$ while $d(x_c, y_c) = n/2 - 1$ or $n - 1$. Since y is not on C we must have $d(x_c, y_c) = n/2 - 1$, to avoid the forbidden parts of C. Also assume, $x_c \neq u_c$. Then, since y_c must be between v_c and w_c (on the shorter part of C and since $d(y, C) = n/2 + 1$, we easily get $K_4 - x \subseteq D_n(G_i)$, which contradicts Lemma 6. Moreover, by the same argument it follows that all vertices of C, except v_c and possibly u_c, w_c, t_1, t_2, t_3, t_4 are of degree 2. In particular, x_c coincides with u_c (or w_c) in which case $d(x, C) = 1$, or t_3 (or t_4) in which case $d(x, C) = n/2 - 1$. The following facts can be easily verified:

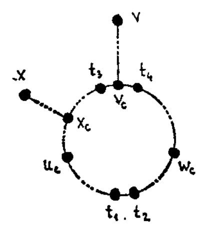

Fig. 7

(18) $\quad l(u_c)$ or $l(w_c) = 0$;

(19) $\quad l(t_3)$ or $l(t_4) \geq n/2 - 1$, implies $l(u_c) = 1(w_c) = 0$;

(20) $\quad l(t_3)$ or $l(t_4) < n/2 - 1$.

From (18)-(20), it follows that each vertex of G_i at distance $n/2$ from the cycle contributes to the appearance of at most one triangle in $D_n(G_i)$. On the other hand $D_n(G_i) = L(G_j)$, where G_j, as can be easily deduced, is a component with the same girth as G_i. Due to the minimally restriction posed in the choice of G_i, it follows that each vertex of degree 3 in G contributes to the appearance of just one triangle of $D_n(G_i)$. Thus, for each vertex x of C, either $l(x) = 0$

or $l(x) \geq n/2$. Suppose now $l(t_3) \geq n/2$. Then $l(t_1) = 0$ (t_3 and its appended path is considered similarly as v_c and path $v_c - v$). Applying (19), it follows that, besides $l(t_3)$ and $l(v_c)$, only $l(t_2)$ may be different from 0. In this case we can easily show, by constructing $D_n(G_i)$, that $C_{2n} \subseteq D_n(G_i)$, which contradicts Lemma 2. If $l(u_c) \geq n/2$, then by arguments similar to those above; it follows that only u_c and v_c can have degrees equal to 2. Thus, G_i consists of a cycle C and at most two hanging paths of length not less than $n/2$ which meet the cycle C in vertices at distances 1 or $n/2-1$. Now, except for $n = 4$, we could always find $K_{1,3} \subseteq D_n(G_i)$. By treating $n = 4$ separately, which is by no means a problem, we finish the proof. □

Lemma 20. *The components of* G *which are not cycles and which have their girth greater than* $2n - 1$ *possibly exist if their girth is equal* $3n - 2$ *or* $3n$.

Proof. Suppose G_i is a component of G with $g(G_i) > 2n - 1$ and $g(G_i) \neq 3n - 2, 3n$. Since G_i is not a cycle we can find four vertices v, v_1, v_2, v_3 of G_i such that $d(v, v_i) = n$ and $d(v_i, v_j) \neq n$ (note, one vertex out of v_1, v_2, v_3 is at distance 1 from C. The latter implies $K_{1,3} \subseteq D_n(G_i)$, a contradiction. □

Lemma. 21. *G has no components of girth* $3n$.

Proof. Suppose the contrary. Then, there exists a component, say G_i, equal to $C_{3n}(k_1, \ldots, k_{3n})$ where not all k's are equal to 0. If for some s $k_s > [n/2]$, then we easily get $C+6 \subseteq D_n(G_i)$, a contradiction by Lemma 2. Thus $k_s \leq [n/2]$ for any s. Let $U = \{u_1, \ldots, u_{3n}\}$ be the vertex set of C (u_s and u_{s+1} are adjacent), while $V = \{v_1, \ldots, v_m\}$ are the remaining vertices of G_i. Then each triplet of vertices u_s, u_{s+n}, u_{s+2n} forms a triangle in $D_n(G_i)$, denoted by T_s. Any vertex v_t, if regarded in $D_n(G_i)$, is adjacent to just two vertices from U, each of them belonging to different triangles (note that n is odd, while $k_s \leq [n/2]$). Vertices from V cannot be adjacent in $D_n(G_i)$, since otherwise we have $K_{1,3} \subseteq D_n(G_i)$. Similarly, each vertex u_s may be adjacent to at most one vertex v_t. Next, let $D_n(G_i) = L(G_j)$ for some j. Since G_j is unicyclic and trianglefree, there must exist in $D_n(G_i)$ a cycle of length greater then 3. Then, in G_i, the following sequence of vertices (indices are ignored) corresponds to the cycle mentioned: $vuuvuu \ldots v$ (the first and the last member of the sequence correspond to the same vertex). If the number of occurrences of u in pairs is less than n, then the vertices of some triangle T_s are not taken into account in the sequence above. Therefore it follows that $L(G_j)$ contains two disjoint cycles, one of which is a triangle. In turn, this implies that G_j contains a vertex of degree 3 outside a cycle, a contradiction by Lemma 7. So we immediately get $G_j = S$, where for brevity we put $S = C_{3n}(1, 0, 0, 1, 0, 0, \ldots, 1, 0, 0)$. Since $D_n(G_i) = L(G_j)$ implies $G_j = S$, then we must have $G_i = S$. Now, if $n \equiv 1 \pmod 3$ (or $n \equiv 2 \pmod 3$), we get that some vertex from U has two neighbors in V (two vertices from V are adjacent). Thus, $n \equiv 0 \pmod 3$. If so, observe the vertices u_1, u_{n+1}, u_{2n+1} chosen so that their degrees are 3. These vertices induce in $D_n(G_i)$ an isolated triangle, a contradiction. □

Next we will investigate only the components of G with the girth $3n - 2$.

LEMMA 22. *If G_i is a component of G with $g(G_i) = 3n-2$, then $G_i = C_{3n-2}(k_1, \dots, k_{3n-2})$, where $k_s \leq 1$.*

Proof. Clearly, we only need to prove that $k_s \leq 1$ for all s. If $k_s > 1$ for some s, then $D_n(G_i)$ contains, among others, two cycles none of which is a triangle. This is a contradiction since all components of G are unicyclic. □

Now, let $\mathcal{H} = \{H | H = C_{3n-2}(k_1, \dots, k_{3n-2})$, where $k_s \leq 1\}$. If $H \in \mathcal{H}$, let $W(H) = \{w | w$ is a vertex of H with $\deg w = 3\}$. Next, let $\varphi := L^{-1} \circ D_n$ provided that L^{-1} does not produce isolated vertices.

LEMMA 23. *Let $H \in \mathcal{H}$. Then $\varphi(H) \in \mathcal{H}$, if the distance between any two vertices from $W(H)$ is not equal to $n-2$.*

Proof. We first note that $d(w_1, w_2) \neq n-2$ for any $w_1, w_2 \in W(H)$, since otherwise $D_n(G_i)$ has at least two cycles none of which is a triangle. On the other hand, if $d(w_1, w_2) \neq n-2$ for all vertex pairs from $W(H)$, we have $\varphi(H) \in \mathcal{H}$. □

Clearly, $H \in \mathcal{H}$ generates a fundamental solution to (1), if and only if $\varphi^p(H) \in \mathcal{H}$ for every $p \geq 0$. To end this, we first examine the effect of changing the distance between any two vertices from $W(H)$, going from H to $\varphi(H)$.

LEMMA 24. *Suppose u and v are vertices of a cycle C of length $3n-2$ with n odd. We then have $d_{D_n(C)}(u,v) = f(d_c(u,v))$ where f is the following function:*

$$f(x) = \begin{cases} 3x/2, & x \text{ is even and } 1 \leq x \leq n-1; \\ 3(n-x)/2 - 1, & x \text{ is odd and } 1 \leq x \leq n-1; \\ 3(n - x/2) - 2, & x \text{ is even and } n \leq x \leq 3(n-1)/2; \\ 3(x-n)/2 + 1, & x \text{ is odd and } n \leq x \leq 3(n-1)/2. \end{cases}$$

Proof. Clearly, $d_{D_n(C)}(u,v) = \min(p, 3n-2-p)$, where p is the smallest nonnegative integer such that $pn = q(3n-2) + x$ $(q \geq 0)$ and $x = d_c(u,v)$. Thus, we have $p = \min(3q - (2q-x)/n))$. Since $r = (2q-x)/n$ is an integer, we get $q = (nr+x)/2$, implying $p = \min(3(nr+x)/2 - r)$, where r is an integer not less than $-x/n$. Next, if x is even, then $r = 0$, while for x being odd $r = \pm 1$, depending on the ratio of x and n. So (21) easily follows. □

Assume now $H \in \mathcal{H}$ implies $\varphi(H) \in \mathcal{H}$. If so, define a mapping from $W(H)$ onto $W(\varphi(H))$ as follows: to each $w \in W(H)$, there corresponds a $w' \in W(\varphi(H))$ such that whenever a hanging edge at w is deleted (which sets H to H^*), then a hanging edge at w', if deleted, gives $\varphi(H^*)$.

LEMMA 25. *Under the above assumptions, if $w_1, w_2 \in W(H)$, then*

$$d_{\varphi(H)}(w_1, w_2) = f(d_H(w_1, w_2)), \tag{22}$$

where f is given by (21).

Proof. Without loss in generality, let $W(H) = \{w_1, w_2\}$. To make it easier, observe Fig. 8, where $d(w_1, a_s) = d(w_2, b_s) = n-1$ $(s = 1,2)$, while c_2, c_2 and d_1, d_2 replace the $a's$ and $b's$ in order to avoid the effects of permuting their indices. Following Fig. 8, we get: $d_{\varphi(H)}(w_1, w_2) = d_{Dn(H)}(c_s d_s)$ $(s = 1,2)$, while on the other hand $d_{D_n(H)}(c_s, d_s) = f(d_H(a_t, b_t)) = f(d_H(w_1, w_2))$ $(s,t = 1,2)$. Now the Lemma easily follows. □

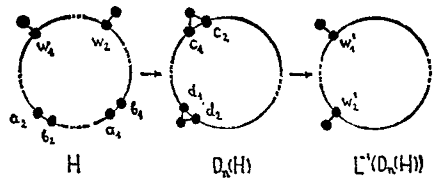

Fig. 8.

2. Main result. Collecting all the conclusions proven thus far in the lemmas, we arrive at our main result.

THEOREM. *The graph equation $L(G) = D_n(G)$ has, as the generators of the fundamental solutions, just the following graphs:*

1° $H = C_k$, *for* $k > 2n$ *and* $(k, n) = 1$*; the period of this graph is* 1.

2° $H = C_5(1, 1, 1, 1, 1)$ *only for* $n = 3$*; the period of this graph is* 1.

3° $H = C_{3n-2}(k_1, \dots, k_{3n-2})$ *provided that:*

a) *n is odd ;*

b) $k_i \leq 1$ *for all* $i = 1, \dots, 3n - 2$*;*

c) *if* u_i, u_j $(u_i \neq u_j)$ *are the vertices of the cycle for which* $k_i, k_j \neq 0$*, then* $d(u_i, u_j) \notin \{f^p(n-2) | p \geq 0\}$*, where f is in fact the permutation given by* (21)*, or in other words* $d(u_i, u_j)$ *does not belong to the cycle of f that contains* $n - 2$.

The period of H in the case 3 is an open question.

Remark. It follows from the theorem above that we can find a general solution for any particular n. The only inconvenience is that for an arbitrary n we don't know the period of some generator, i.e., we don't know in advance the number of components of some fundamental solution.

In order to illustrate this theorem, we deduce the solution for $n = 3$. In this case the general solution consists of the components of the following three types: C_k ($k > 6$ and $k \not\equiv 0 \pmod 3$), $C_5\,(1, 1, 1, 1, 1)$ and $C_7\,(1, 0, 0, 0, 0, 0, 0)$.

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