

**EXTRAPOLATION OF MOVING AVERAGE AND
 AUTOREGRESSIVE PROCESSES WHEN THE ENTIRE
 PAST OF THE PROCESSES IS KNOWN**

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1. Introduction. Let $X(t) = (X_1(t), \dots, X_n(t))$, $t \in R$, be a multidimensional and wide sense stationary random process with the mean value zero, the correlation matrix $\|B_{jk}^X(\tau)\|$ and the spectral process $Z^X(\lambda) = (Z_1^X(\lambda), \dots, Z_n^X(\lambda))$ $\lambda \in R$. Suppose that all elements $B_{jk}(\tau)$ of the correlation matrix $\|B_{jk}(\tau)\|$ fall off sufficiently rapidly at infinity so that they can be represented in the form:

$$B_{jk}^X(\lambda) = \int_{-\infty}^{+\infty} e^{i\tau\lambda} f_{jk}^X(\lambda) d\lambda, \quad i, k = 1, 2, \dots, n \quad (1.1)$$

(i.e. there exist the spectral densities $f_{jk}^X(\lambda)$).

Let H denote the Hilbert space of all second-order random functions, where a scalar product is defined by $(\xi, \eta) = E\xi\bar{\eta}$, and let $H(X)$ be the Hilbert space generated by $\{X_k(t) : t \in R, k = 1, 2, \dots, n\}$. Let $L^2(F)$ be the space of functions $\Phi(\lambda) = (\Phi_1(\lambda), \dots, \Phi_n(\lambda))$ satisfying the following condition:

$$\sum_{k=1}^n \sum_{j=1}^n \int_{-\infty}^{+\infty} \Phi_k(\lambda) \overline{\Phi_j(\lambda)} f_{j,k}^X(\lambda) d\lambda < \infty \quad (1.2)$$

The space $L^2(F)$ becomes a Hilbert space if we define the scalar product $(\Phi(\lambda), \Psi(\lambda))$ by:

$$(\Phi(\lambda), \Psi(\lambda)) = \int_{-\infty}^{+\infty} \sum_{k=1}^n \sum_{j=1}^n \Phi_k(\lambda) \overline{\Psi_j(\lambda)} f_{j,k}^X(\lambda) d\lambda < \infty \quad (1.3)$$

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A random function ξ belongs to $H(X)$ if and only if there exists $\Phi(\lambda) \in L^2(F)$ so that

$$\xi = \sum_{k=1}^n \int_{-\infty}^{+\infty} \Phi_k(\lambda) dZ_k(\lambda) \quad (1.4)$$

The mapping of $H(X)$ to $L^2(F)$ given by $\xi = \sum_{k=1}^n \int \Phi_k(\lambda) dZ_k(\lambda) \rightarrow (\Phi_1(\lambda), \dots, \Phi_n(\lambda))$ is an isometry.

A stationary random process $X(t)$, $t \in R$, is said to be a process with a rational spectrum if all the functions $f_{jk}^X(\lambda)$ are rational functions, i.e. $f_{jk}(\lambda) = Q_{jk}(\lambda)/P_{jk}(\lambda)$, where $P_{jk}(\lambda)$ and $Q_{jk}(\lambda)$ are polynomials in λ . The spectral densities $f_{jj}(\lambda)$ can be represented in the form:

$$B_0|\lambda^M + B_1\lambda^{M-1} + \dots + B_N|^2|\lambda^P + A_1\lambda^{P-1} + \dots + A_P|^{-2} \quad (1.5)$$

where $B_0 > 0$ and $M < P$. Denote the determinant of the spectral matrix $\|f_{jk}(\lambda)\|$ by $D(\lambda) (= Q(\lambda)/P(\lambda))$. Let $2N_{jj}$, $2(N_{jj} - m_j)$, $2K$, $2L$ be the degrees of $P_{jj}(\lambda)$, $Q_{jj}(\lambda)$, $P(\lambda)$, $Q(\lambda)$ respectively. Then $2K - 2L \geq 2 \sum_{k=1}^n m_k$.

A stationary random process with a rational spectrum is nonsingular if the determinant $D(\lambda)$ has no real zeros and $2K - 2L \geq 2 \sum_{k=1}^n m_k$. The polynomial $Q(\lambda)$ can be represented in the form $B(\lambda - \theta_1) \dots (\lambda - \theta_L)(\lambda - \bar{\theta}_1) \dots (\lambda - \bar{\theta}_L)$ where $\Im \theta_i > 0$, $i = 1, 2, \dots, L$.

Now, suppose we know the values $X_k(s)$, $s \leq t$, $k = 1, 2, \dots, n$. The simplest problem of linear extrapolation of a stationary random process $X(t)$ is the following: Find a random variable

$$\tilde{X}_1(t, T) = \sum_{k=1}^n \int_{-\infty}^{+\infty} \Phi_k(\lambda) dZ_k(\lambda) \in H(X)$$

which is the best approximation of $X_1(t + \tau)$, depending linearly on $X_k(s)$, $s \leq t$, $k = 1, 2, \dots, n$. As an index of approximation quality we shall use the mean square error. We shall call $\tilde{X}_1(t, \tau)$ the linear least-squares estimator of $X_1(t + \tau)$. We see that $X_1(t, \tau)$ is the projection of $X_1(t + \tau)$ into the smallest Hilbert space $H(X, t)$ spanned by $\{X_k(s), s \leq t, k = 1, 2, \dots, n\}$.

In the paper [7] the following theorem was proved:

THEOREM 1. (Yaglom): *If $X(t) = (X_1(t), \dots, X_n(t))$ is a nonsingular stationary random process with a rational spectrum, then the linear least-squares estimator of $X_1(t + \tau)$ is given by:*

$$\tilde{X}_1(t, \tau) = \sum_{k=1}^n \left\{ \sum_{j=0}^{m_k-1} w_k^{(j)}(t) + \int_0^\infty w_k(s, \tau) X_k(t - s) ds \right\} \quad (1.6)$$

where the coefficients $w_k^{(j)}(\tau)$ and the functions $w_k(s, \tau)$ can be found (in a unique way) from the condition: the functions

$$\Phi_k(\lambda) = \sum_{j=0}^{m_k-1} w_k^{(j)}(\tau)(i\lambda)^j + \int_0^\infty e^{-is\lambda} w_k(s, \tau) ds \quad (1.7)$$

have the form

$$\Phi_k(\lambda) = \omega_k(\lambda) \{(\lambda - \theta_1) \dots (\lambda - \theta_l)\}^{-1}, \quad k = 1, 2, \dots, n. \quad (1.8)$$

where $\omega_k(\lambda)$ is a polynomial of degree $L + m_k - 1$ (but not of a greater degree) such that the functions

$$\Psi_k(\lambda) = (e^{i\tau\lambda} - \Phi_1(\lambda))f_{1k}(\lambda) - \sum_{j=2}^n \Phi_j(\lambda)f_{jk}(\lambda) \quad (1.9)$$

are analytic in the upper half plane.

The functions $\Phi_k(\lambda)$, $k = 1, 2, \dots, n$, will be called the spectral characteristic for extrapolation of the random process $X(t)$ at the point $t + \tau$.

2. Moving average and autoregressive processes

Let $X(t) = (X_1(t), \dots, X_n(t))$ and $Y(t) = (Y_1(t), \dots, Y_n(t))$ be two multidimensional stationary random processes satisfying the following equation

$$Y(t) = \sum_{\nu=0}^N a_\nu X(t - \nu\theta), \quad a_0 = 1, \quad \theta > 0 \quad (2.1)$$

The process $Y(t)$ is called the moving average process of order N associated with the process $X(t)$. We shall denote it by $MA_X(N, \theta)$. The process $X(t)$ is called the autoregressive process of order N associated with the process $Y(t)$. We shall denote it by $AR_Y(N, \theta)$.

If the roots of

$$t^N + a_1 t^N + \dots + a_N = 0 \quad (2.2)$$

are less than one in absolute value, then the following equations also hold:

$$X_k(t) = \sum_{\nu=0}^{\infty} c_\nu Y_k(t - \nu\theta), \quad k = 1, 2, \dots, n. \quad (2.3)$$

The series on the right side of (2.3) converges in quadratic mean and the coefficients c_ν satisfy the homogeneous difference equation

$$a_0 c_k + a_1 c_{k-1} + \dots + a_N c_{k-N} = 0, \quad k > N \quad (2.4)$$

and the initial conditions

$$\begin{aligned} a_0 &= 1 \\ a_0 c_k + a_1 c_{k-1} + \dots + a_k c_0 &= 0, \quad 0 < k < N. \end{aligned} \quad (2.5)$$

THEOREM 2. Let $X(t)$ and $Y(t)$ be a multidimensional stationary random processes with spectral processes $(Z_1^X(\lambda), \dots, Z_n^X(\lambda))$ and $(Z_1^Y(\lambda), \dots, Z_n^Y(\lambda))$ and let $X(t)$ and $Y(t)$ satisfy (2.1). Then:

$$\text{a)} \quad dZ_k^Y(\lambda) = \left\{ \sum_{\nu=0}^N a_\nu e^{-i\lambda\nu\theta} \right\} dZ_k^X(\lambda), \quad k = 1, 2, \dots, n \quad (2.6)$$

$$\text{b)} \quad f_{jk}(\lambda) = \left| \sum_{\nu=0}^N a_\nu e^{-i\nu\theta\lambda} \right|^2 f_{jk}(\lambda), \quad j, k = 1, 2, \dots, n \quad (2.7)$$

c) If the roots of (2.2) are less than one in absolute value then

$$H(X, t) = H(Y, t). \quad (2.8)$$

Proof: The equation (2.6) follows from

$$\int_{-\infty}^{+\infty} e^{it\lambda} dZ_k^Y(\lambda) = Y(t) = \sum_{\nu=0}^N a_\nu \int_{-\infty}^{+\infty} e^{i(t-\nu\theta)\lambda} dZ_k^X(\lambda) = \int_{-\infty}^{+\infty} e^{it\lambda} \sum_{\nu=0}^N a_\nu e^{-i\nu\theta\lambda} dZ_k^X(\lambda).$$

The equation (2.7) follows from

$$\begin{aligned} & \int_{-\infty}^{+\infty} e^{i\tau\lambda} f_{jk}^Y(\lambda) d\lambda = B_{jk}^Y(\tau) = EY_j(t + \tau) \overline{Y_k(t)} = \\ & = E \sum_{\nu=0}^N a_\nu X_j(t + \tau - \nu\theta) \sum_{\mu=0}^N \bar{a}_\mu \overline{X_k(t - \mu\theta)} = \sum_{\nu=0}^N \sum_{\mu=0}^N a_\nu \bar{a}_\mu B_{jk}^X(\tau - (\nu - \mu)\theta) \\ & = \sum_{\nu=0}^N \sum_{\mu=0}^N a_\nu \bar{a}_\mu \int_{-\infty}^{+\infty} e^{i\tau\lambda} e^{i(\nu-\mu)\theta\lambda} f_{jk}^X(\lambda) d\lambda = \int_{-\infty}^{+\infty} e^{i\tau\lambda} \left| \sum_{\nu=0}^N a_\nu e^{-i\nu\theta\lambda} \right|^2 f_{jk}^X(\lambda) d\lambda. \end{aligned}$$

The result (2.8) follows from (2.1) and (2.3).

In this paper, we shall solve the simplest extrapolation problem for the moving average and autoregressive processes associated with a nonsingular stationary process with a rational spectrum.

3. Linear extrapolation of moving average processes

THEOREM 3. Let $X(s)$ and $Y(s)$ be two multidimensional stationary processes satisfying (2.1) and let the roots of the equation (2.2) be less than one in absolute value. If $\Phi_{k,\tau}^X(\lambda)$ and $\Phi_{k,\tau}^Y(\lambda)$ are the spectral characteristics for extrapolation of the processes $X(s)$ and $Y(s)$ at the point $t + \tau$ ($\tau > 0$), then:

a) If τ/θ is not an integer and $[\tau/\theta] = S < N$, then:

$$\Phi_{1,\tau}^Y(\lambda) = \left\{ \sum_{\nu=0}^S a_\nu \Phi_{1,\tau-\nu\theta}^X(\lambda) + \sum_{\nu=S+1}^N a_\nu e^{i(\tau-\nu\theta)\lambda} \right\} \left\{ \sum_{\nu=0}^N a_\nu e^{-i\nu\theta\lambda} \right\}^{-1} \quad (3.1)$$

$$\Phi_{k,\tau}^Y(\lambda) = \sum_{\nu=0}^S a_\nu \Phi_{1,\tau-\nu\theta}^X(\lambda) \left\{ \sum_{\nu=0}^N a_\nu e^{-i\nu\theta\lambda} \right\}^{-1}, \quad k = 2, \dots, n. \quad (3.2)$$

b) If $\tau = S\theta$ and $S \leq N$, it is only necessary to replace S by $S-1$ in the equations (3.1) and (3.2).

c) If τ/θ is not an integer and $S \geq N$ or if $\tau = S\theta$ and $S > N$, then:

$$\Phi_{k,\tau}^Y(\lambda) = \sum_{\nu=0}^N a_\nu \Phi_{1,\tau-\nu\theta}^X(\lambda) \left\{ \sum_{\nu=0}^N a_\nu e^{-i\nu\theta\lambda} \right\}^{-1}, \quad k = 2, \dots, n. \quad (3.3)$$

Proof. a) If τ/θ is not an integer and $S < N$, we have $\tau - \nu\theta > 0$ for $\nu = 0, 1, 2, \dots, S$ and $\tau - \nu\theta \leq 0$ for $\nu = S+1, \dots, N$.

It follows from (2.8) that

$$\begin{aligned} \tilde{Y}_1(t, \tau) &= \text{Proj}_{H(Y,t)} Y_1(t+\tau) = \sum_{\nu=0}^N \text{Proj}_{H(X,t)} X_1(t+\tau-\nu\theta) = \\ &= \sum_{\nu=0}^S a_\nu \tilde{X}_1(t, \tau-\nu\theta) + \sum_{\nu=S+1}^N a_\nu X_1(t+\tau-\nu\theta). \end{aligned}$$

If we apply (1.6) and (2.6), we obtain

$$\begin{aligned} \tilde{Y}_1(t, \tau) &= \int_{-\infty}^{+\infty} e^{it\lambda} \left\{ \sum_{\nu=0}^S a_\nu \Phi_{k,\tau-\nu\theta}^X(\lambda) + \sum_{\nu=S+1}^N a_\nu e^{i(\tau-\nu\theta)\lambda} \right\} \left\{ \sum_{\nu=0}^N a_\nu e^{-i\nu\theta\lambda} \right\}^{-1} dZ_1^X(\lambda) + \\ &\quad + \sum_{k=2}^n \int_{-\infty}^{+\infty} e^{it\lambda} \sum_{\nu=0}^S a_\nu \Phi_{k,\tau-\nu\theta}^X(\lambda) \left\{ \sum_{\nu=0}^N a_\nu e^{-i\nu\theta\lambda} \right\}^{-1} dZ_k^X(\lambda) \end{aligned}$$

and the desired result follows.

b) and c) The proof is similar.

COROLLARY: If $X(s), s \in R$, is a nonsingular stationary random process with rational spectrum, then the spectral characteristics $\Phi_{k,\tau}^Y(\lambda)$, $k = 1, 2, \dots, n$, can be represented in the form:

a) If τ/θ is not an integer and $S < N$ then:

$$\Phi_{1,\tau}^Y(\lambda) R_1(\lambda) \sum_{\nu=0}^{\infty} c_\nu^{(1)} e^{-i\nu\theta\lambda} + \sum_{\nu=S+1}^{\infty} c_\nu^{(2)} e^{i(\tau-\nu\theta)\lambda} \quad (3.4)$$

$$\Phi_{k,\tau}^Y(\lambda) R_k(\lambda) \sum_{\nu=0}^{\infty} c_\nu e^{-i\nu\theta\lambda}, \quad k = 2, \dots, n. \quad (3.5)$$

b) If $\tau = s\theta$ and $S \leq N$, it is necessary to replace S by $S - 1$ in the formulae (3.4) and (3.5).

c) If τ/θ is not an integer and $S \geq N$ or if $\tau = S\theta$ and $S > N$ then:

$$\Phi_{k,\tau}^Y(\lambda) = R_k(\lambda) \sum_{\nu=0}^{\infty} c_{\nu} e^{-i\nu\theta\lambda}, \quad k = 1, 2, \dots, n. \quad (3.6)$$

The functions $R_k(\lambda) = \sum_{\nu} a_{\nu} \Phi_{k,\tau-\nu\theta}(\lambda)$ are rational functions.

Proof: If α_{ν} , $\nu = 1, 2, \dots, n$, are the roots of the equation (2.2), we have

$$\left\{ \sum_{\nu=0}^N a_{\nu} e^{-i\nu\theta\lambda} \right\}^{-1} = \prod_{\nu=0}^N (1 - \alpha_{\nu} e^{-i\nu\theta\lambda})^{-1} = \sum_{\nu=0}^{\infty} c_{\nu} e^{-i\nu\theta\lambda}$$

and the proof follows.

THEOREM 4. Let $X(s)$ and $Y(s)$ satisfy the assumptions of Theorem 3, and let $X(s)$ be a nonsingular stationary process with rational spectrum. Suppose we know the values $Y_k(s)$, $s \leq t$, $k = 1, 2, \dots, n$. If τ/θ is not an integer and $S < N$, then the linear least-squares estimator of $Y_1(t + \tau)$ is given by:

$$\begin{aligned} \tilde{Y}_1(t, \tau) = & \sum_{k=1}^n \left\{ \sum_{j=-m_k}^{m_k-1} \sum_{\alpha=0}^{\infty} A_{k,\alpha}^{(j)} Y_k(j)(t - \alpha\theta) + \int_0^{\infty} B_k(u) Y_k(t-u) du \right\} + \\ & + \sum_{\mu=0}^{\infty} D_{\mu} Y_1(t + \tau - (S+1)\theta - \mu\theta), \end{aligned} \quad (3.7)$$

where

$$\begin{aligned} A_{k,\alpha}^{(j)} &= c_{\alpha} \sum_{\nu=0}^S a_{\nu} w_k^{(j)}(\tau - \nu\theta), \\ B_k(u) &= \sum_{\alpha=0}^{\infty} c_{\alpha} D_k(u - \alpha\theta, \tau), \end{aligned} \quad (3.8)$$

$$D_k(u - \alpha\theta, \tau) = \begin{cases} 0 & 0 \leq u < \alpha\theta \\ \sum_{\nu=0}^S a_{\nu} w_k(u - \alpha\theta, \tau - \nu\theta) & u \geq \alpha\theta \end{cases}, \quad \alpha = 0, 1, \dots, \quad (3.10)$$

$$D_{\mu} = \sum_{\nu=S+1}^N a_{\mu} F(\mu, \nu) \quad (3.11)$$

$$F(\mu, \nu) = \begin{cases} 0 & 0 \leq \mu < \nu - S - 1 \\ c_{\mu-\nu+S+1} & \mu \geq \nu - S - 1 \end{cases}, \quad \nu = S + 1, \dots, N \quad (3.12)$$

where the coefficients c_{ν} were given by (2.4) and (2.5) and $w_k^{(j)}$, $w_k(s, \tau)$ were determined by Theorem 1.

Proof. Using the formulae (1.6) and (2.3) we obtain

$$\begin{aligned}
\tilde{Y}_1(t, \tau) &= \sum_{\nu=0}^N a_\nu \tilde{X}_1(t, \tau - \nu\theta) + \sum_{\nu=S+1}^N a_\nu X(t + \tau - \nu\theta) = \\
&= \sum_{\nu=0}^N a_\nu \left\{ \sum_{k=1}^n \left[\sum_{j=0}^{m_k-1} w_k^{(j)}(\tau - \nu\theta) X_k^{(j)}(t) + \int_0^\infty w_k(s, \tau - \nu\theta) X_k(t-s) ds \right] \right\} + \\
&+ \sum_{\nu=S+1}^N a_\nu X_1(t + \tau - \nu\theta) = \sum_{k=1}^n \sum_{\alpha=0}^\infty c_\alpha \left\{ \sum_{j=0}^{m_k-1} \sum_{\nu=0}^S a_\nu w_k^{(j)}(\tau - \nu\theta) Y_k^{(j)}(t - \alpha\theta) + \right. \\
&\quad \left. + \int_0^\infty \sum_{\nu=0}^S a_\nu w_k(s, \tau - \nu\theta) Y_k(t-s - \alpha\theta) ds \right\} + \\
&+ \sum_{\alpha=0}^\infty \left\{ c_\alpha \sum_{\nu=S+1}^N a_\nu a_\nu \right\} Y_1(t + \tau - \nu\theta - \alpha\theta),
\end{aligned}$$

and (3.7) follows from (3.8) – (3.12) and

$$\begin{aligned}
&\sum_{\alpha=0}^\infty c_\alpha \int_0^\infty \sum_{\nu=0}^S a_\nu w_k(s, \tau - \nu\theta) Y_k(t-s - \tau - \alpha\theta) ds = \\
&= \sum_{\alpha=0}^\infty c_\nu \int_{\alpha\theta}^\infty \sum_{\nu=0}^S a_\nu w_k(u - \alpha\theta, \tau - \nu\theta) \cdot Y_k(t-u) du = \\
&= \sum_{\alpha=0}^\infty c_\nu \int_0^\infty D_k(u - \alpha\theta, \tau) Y_k(t-u) du = \int_0^\infty B_k(u) Y_k(t-u) du, \\
&\sum_{\nu=S+1}^N a_\nu \sum_{\alpha=0}^\infty c_\alpha Y_1(t + \tau - \nu\theta - \alpha\theta) Y_k(t-u) du = \\
&= \sum_{\nu=S+1}^N a_\nu \sum_{\mu=\nu-S-1}^\infty c_{\mu-\nu+S+1} Y_1(t + \tau - (S+1)\theta - \mu\theta) = \\
&= \sum_{\nu=S+1}^N a_\nu F(\mu, \nu) Y_1(t + \tau - (S+1)\theta - \mu\theta) = \sum_{\mu=0}^\infty D_\mu Y_1(t + \tau - (S+1)\theta - \mu\theta).
\end{aligned}$$

THEOREM 5. If $T = S\theta$ and $S \leq N$, we can obtain $\tilde{Y}_1(t, \tau)$ if we replace S by $S-1$ in the formulae (3.7) – (3.12).

If τ/θ is not an integer and $S \geq N$, or if $\tau = S\theta$ and $S > N$, then $\tilde{Y}_1(t, \tau)$ is given by:

$$\tilde{Y}(t, \tau) = \sum_{k=1}^n \left\{ \sum_{j=0}^{m_k-1} \sum_{\alpha=0}^\infty A_{k,\alpha}^{(j)} Y_k^{(j)}(t - \alpha\theta) + \int_0^\infty B_k(u) Y_k(t-u) du \right\} \quad (3.13)$$

where

$$A_{k,\alpha}^{(j)} = c_\alpha \sum_{\nu=0}^N a_\nu w_k^{(j)}(\tau - \nu\theta), \quad (3.14)$$

$B_k(u)$ is given by (3.9) and

$$D_k(u - \alpha\theta, \tau) = \begin{cases} 0 & 0 \leq u < \alpha\theta \\ \sum_{\nu=0}^S a_\nu w_k(u - \alpha\theta, \tau - \nu\theta) & u \geq \alpha\theta \end{cases}, \quad (3.15)$$

$\alpha = 0, 1, \dots$

4. Linear extrapolation of autoregressive processes

THEOREM 6. Let $X(s)$ and $Y(s)$ satisfy the assumptions of Theorem 3. Then:

a) If $\tau\theta$ is not an integer and $[\tau/\theta] = S$, then

$$\Phi_{1,\tau}^X(\lambda) = \sum_{\nu=0}^S c_\nu \Phi_{1,\tau-\nu\theta}^Y(\lambda) \sum_{\nu=0}^N a_\nu e^{-i\nu\theta\lambda} + \sum_{\mu=S+1}^{S+N} D_\mu e^{i(\tau-\mu\theta)\lambda} \quad (4.1)$$

$$D_\mu = \sum_{\nu=0}^{\mu-S-1} a_\nu c_{\mu-\nu}, \quad \mu = S+1, \dots, S+N. \quad (4.2)$$

$$\Phi_{k,\tau}^X(\lambda) = \sum_{\nu=0}^S c_\nu \Phi_{k,\tau-\nu\theta}^Y(\lambda) \sum_{\nu=0}^N a_\nu e^{-i\nu\theta\lambda}, \quad k = 2, \dots, n. \quad (4.3)$$

b) If $\tau = S\theta$, then it is necessary to replace S by $S-1$ in the equations (4.1) – (4.3).

Proof. a) It follows from (2.3), (2.6) and (2.8) that

$$\begin{aligned} \tilde{X}_1(t, \tau) &= \text{Proj}_{H(X,t)} X_1(t + \tau) = \text{Proj}_{H(Y,t)} \sum_{\nu=0}^{\infty} c_\nu Y_1(t + \tau - \nu\theta) \\ &= \sum_{\nu=0}^S c_\nu \tilde{Y}_1(t, \tau - \nu\theta) + \sum_{\nu=S+1}^{\infty} c_\nu Y_1(t + \tau - \nu\theta) \\ &= \sum_{\nu=0}^S c_\nu \sum_{k=1}^n \int_{-\infty}^{+\infty} e^{it\lambda} \Phi_{k,\tau-\nu\theta}(\lambda) dZ_k^Y(\lambda) + \sum_{\nu=S+1}^{\infty} c_\nu \int_{-\infty}^{+\infty} e^{i(t+\tau-\nu\theta)\lambda} dZ_1^Y(\lambda) \\ &= \int_{-\infty}^{+\infty} e^{it\lambda} \left\{ \sum_{\nu=0}^S c_\nu \Phi_{1,\tau-\nu\theta}(\lambda) + \sum_{\nu=S+1}^{\infty} c_\nu e^{i(\tau-\nu\theta)\lambda} \right\} \left\{ \sum_{\nu=0}^N a_\nu e^{-i\nu\theta\lambda} dZ_1^X(\lambda) \right. \\ &\quad \left. + \sum_{k=2}^n \int_{-\infty}^{+\infty} e^{it\lambda} \sum_{\nu=0}^S c_\nu \Phi_{k,\tau-\nu\theta}(\lambda) \sum_{\nu=0}^N a_\nu e^{-i\nu\theta\lambda} dZ_k^X(\lambda) \right\}. \end{aligned}$$

The term $\sum_{\nu=0}^{\infty} c_{\nu} e^{i(\tau-\nu\theta)\lambda} \sum_{\nu=0}^N a_{\nu} e^{-i\nu\theta\lambda}$ can be represented in the form

$$\sum_{\mu=S+1}^{S+N} \sum_{\nu=0}^{\mu-S-1} a_{\nu} c_{\mu-\nu} e^{i(\tau-\mu\theta)\lambda} + \sum_{\mu=S+N+1}^{\infty} \sum_{\nu=0}^N a_{\nu} c_{\mu-\nu} e^{i(\tau-\mu\theta)\lambda}$$

and the desired result follows if we use (2.4) and (2.5).

COROLLARY: *If $Y(s)$, $s \in R$, is a nonsingular stationary process with a rational spectrum and $X = AR_Y(N, \theta)$, then the spectral characteristics $\Phi_{k,\tau}(\lambda)$ can be represented in the form:*

$$\begin{aligned} \Phi_{1,\tau}^X(\lambda = R_1(\lambda) \sum_{\nu=0}^N a_{\nu} e^{-i\nu\theta\lambda} + \sum_{\mu=S'}^{S'+N-1} D_{\mu} e^{i(\tau-\mu\theta)\lambda} \\ \Phi_{k,\tau}^X(\lambda = R_k(\lambda) \sum_{\nu=0}^N a_{\nu} e^{-i\nu\theta\lambda}, \quad k = 2, \dots, n \end{aligned}$$

where $S' = S + 1$ if τ/θ is not an integer and $S' = S$ if $\tau = S\theta$.

The functions $R_k(\lambda)$, $k = 1, 2, \dots, n$, are rational functions.

THEOREM 7. *Let $X(s)$ and $Y(s)$ be two multidimensional stationary processes satisfying (2.1) and let the roots of the equation (2.2) be less than one in absolute value. Let $Y(s)$ be a nonsingular random process with a rational spectrum. Suppose we know the values of the stationary process $X(s) = AR_{Y(s)}(N, \theta)$ for $s \leq t$. Then, the linear least-squares estimator of $X_1(t + \tau)$, $\tau > 0$, is given by:*

a) *If τ/θ is not an integer and $[\tau/\theta] = S$, then:*

$$\begin{aligned} \tilde{X}_1(t, \tau) &= \sum_{k=1}^n \left\{ \sum_{j=0}^{m_k-1} \sum_{\alpha=0}^N B_{k,\alpha}^{(j)} X_k^{(j)}(t - \alpha\theta) + \int_0^\infty B_k(u) X_k(t-u) du \right\} \\ &= \sum_{\mu=0}^{N-1} D_{\mu} X_1(t + \tau - (S + 1) - \mu\theta), \end{aligned} \tag{4.4}$$

where

$$B_{k,\alpha}^{(j)} = a_{\alpha} \sum_{\nu=0}^S c_{\nu} w_k^{(j)}(\tau - \nu\theta), \tag{4.5}$$

$$B_k(u) = \sum_{\alpha=0}^N a_{\alpha} D_k(u - \alpha\theta, \tau), \tag{4.6}$$

$$D_k(u - \alpha\theta, \tau) = \begin{cases} 0 & 0 \leq u < \alpha\theta \\ \sum_{\nu=0}^S c_\nu w_k(u - \alpha\theta, \tau - \nu\theta) & u \geq \alpha\theta \end{cases} \quad (4.7)$$

$$D_\mu = \sum_{\alpha=0}^N a_\alpha F(\mu, \alpha), \quad \mu = 0, 1, \dots, N-1 \quad (4.8)$$

$$F(\mu, \nu) = \begin{cases} 0 & 0 \leq \mu < \alpha \\ c_{S+1+\mu-\alpha} & \mu \geq \alpha \end{cases}, \quad \alpha = 0, 1, 2, \dots, N \quad (4.9)$$

b) If $\tau = S\theta$, it is necessary to replace S by $S-1$ in the equations (4.4) – (4.9).

Proof. a) Since $Y(s)$ is a nonsingular stationary process with a rational spectrum, we have

$$\tilde{Y}_1(t, \tau) = \sum_{k=1}^n \left\{ \sum_{j=0}^{m_k-1} w_k^{(j)}(\tau) Y_k^{(j)}(t) + \int_0^\infty w_k(s, \tau) Y_k(t-s) ds \right\}$$

and

$$\begin{aligned} \tilde{X}_1(t, \tau) &= \sum_{\nu=0}^S c_\nu \left\{ \sum_{k=1}^n \left[\sum_{j=0}^{m_k-1} w_k^{(j)}(\tau - \nu\theta) Y_k^{(j)}(t) + \int_0^\infty w_k(s, \tau - \nu\theta) Y_k(t-s) ds \right] + \right. \\ &\quad \left. + \sum_{\nu=S+1}^\infty c_\nu Y_1(t + \tau - \nu\theta) \right\} = \\ &= \sum_{k=1}^n \left\{ \sum_{j=0}^{m_k-1} \sum_{\alpha=0}^N \left(a_\alpha \sum_{\nu=0}^S c_\nu w_k^{(j)}(\tau - \nu\theta) \right) X_k^{(j)}(t - \alpha\theta) + \right. \\ &\quad \left. + \int_0^\infty \sum_{\nu=0}^S c_\nu w_k(s, \tau - \nu\theta) \sum_{\alpha=0}^N a_\alpha X_k(t-s - \alpha\theta) ds \right\} + \\ &\quad + \sum_{\nu=S+1}^\infty c_\nu \sum_{\alpha=0}^N a_\alpha X_1(t + \tau - \nu\theta - \alpha\theta) \end{aligned}$$

and (4.4) follows from (4.5) – (4.9) and

$$\begin{aligned} \sum_{\alpha=0}^N a_\alpha \int_0^\infty \sum_{\nu=0}^S c_\nu w_k(s, \tau - \nu\theta) X_k(t-s - \alpha\theta) ds &= \sum_{\alpha=0}^N a_\alpha \int_0^\infty \sum_{\nu=0}^S c_\nu w_k(u - \alpha\theta, \tau - \nu\theta) \cdot \\ &\quad \cdot X_k(t-u) du = \sum_{\alpha=0}^N a_\alpha \int_0^\infty D_k(u - \alpha\theta, \tau) X_k(t-u) du = \int_0^\infty B_k(u) X_k(t-u) du, \\ \sum_{\nu=S+1}^\infty c_\nu \sum_{\alpha=0}^N a_\alpha X_1(t + \tau - \nu\theta - \alpha\theta) &= \sum_{\alpha=0}^N \sum_{\mu=\alpha}^\infty c_{S+1+\mu-\alpha} X_1(t + \tau - (S+1)\theta - \mu\theta) \\ &= \sum_{\alpha=0}^N a_\alpha \sum_{\mu=\alpha}^\infty F(\mu, \alpha) X_1(t + \tau - (S+1)\theta - \mu\theta) = \sum_{\mu=0}^{N-1} D_\mu X_1(t + \tau - (S+1)\theta - \mu\theta). \end{aligned}$$

The last equation follows from (2.4).

5. Let us now give some examples.

Example 1. Let $x_1(s)$ and $x_2(s)$ be two independent stationary random processes with the rational spectral densities

$$f_1(\lambda) = c_1(\lambda^2 + \alpha_1^2)^{-1}, \quad f_2(\lambda) = c_2(\lambda^2 + \alpha_2^2)^{-1}, \quad c_1, c_2, \alpha_1, \alpha_2 > 0, \quad \alpha_1 \neq \alpha_2,$$

and let $X_1(s) = x_1(s) + x_2(s)$, $X_2(s) = x_1(s) - x_2(s)$. It is easy to see that the spectral density matrix $\|f_{ij}^k(\lambda)\|_{2 \times 2}$ of the stationary random process $X(s) = (X_1(s), X_2(s))$ is given by

$$\begin{aligned} M^X(\lambda) &= \|f_{ij}^{(X)}(\lambda)\| = \\ &= \begin{vmatrix} c_1(\lambda^2 + \alpha_1^2)^{-1} + c_2(\lambda^2 + \alpha_2^2)^{-1} & c_1(\lambda^2 + \alpha_1^2)^{-1} - c_2(\lambda^2 + \alpha_2^2)^{-1} \\ c_1(\lambda^2 + \alpha_1^2)^{-1} - c_2(\lambda^2 + \alpha_2^2)^{-1} & c_1(\lambda^2 + \alpha_1^2)^{-1} + c_2(\lambda^2 + \alpha_2^2)^{-1} \end{vmatrix} \end{aligned} \quad (5.1)$$

In this case $D(\lambda) = 4c_1c_2(\lambda^2 + \alpha_1^2)^{-1}(\lambda^2 + \alpha_2^2)^{-1}$ has no real zeros, $n = 2$, $2L = 0$, $2K = 4$, $m_1 = m_2 = 1$, $2K - 2L = 2(m_1 + m_2)$, and thus $X(s)$ is a nonsingular stationary random process with a rational spectrum.

Suppose we know the values of the process $X(s)$ for $s \leq t$. Using Theorem 1 we have:

$$\begin{aligned} \Phi_1(\lambda) &= \omega_1(\lambda) = K_1, \quad \Phi_2(\lambda) = \omega_2(\lambda) = K_2, \\ \Psi_1(\lambda) &= (e^{i\tau\lambda} - K_1)f_{11}(\lambda) - K_2f_{21}(\lambda), \\ \Psi_2(\lambda) &= (e^{i\tau\lambda} - K_1)f_{12}(\lambda) - K_2f_{22}(\lambda). \end{aligned}$$

From the fact that the functions $\Psi_1(\lambda)$ and $\Psi_2(\lambda)$ are analytic in the upper half-plane, it follows that

$$2K_1 = 2K_1(T) = e^{-\tau\alpha_1} + e^{-\tau\alpha_2}, \quad 2K_2 = K_2(\tau) = e^{-\tau\alpha_1} - e^{-\tau\alpha_2} \quad (5.2)$$

Then, we have $w_1^{(0)}(\tau) = K_1(\tau)$, $w_2^{(0)}(\tau) = K_2(\tau)$, $w_1(s, \tau) = w_2(s, \tau) = 0$, and

$$2X_1(t, \tau) = (e^{-\tau\alpha_1} + e^{-\tau\alpha_2})X_1(t) + e^{-\tau\alpha_1} - e^{-\tau\alpha_2}X_2(t). \quad (5.3)$$

Example 2. Let $X(s)$ be the process given in the example 1, $Y(s) = X(s) - \beta X(s-1)$, $|\beta| < 1$, and suppose we know the values of the process $X(s)$ for $s \leq t$.

In this case: $N = 1$, $a_0 = 1$, $a_1 = -\beta$, $Y(s) = MA_X(1, \theta)$, $X(s) = \sum_{\alpha=0}^{\infty} \beta^\alpha Y(s - \alpha\theta)$.

a) If $0 < \tau < \theta$ ($0 = S < N = 1$), using the formulae (3.8) – (3.12) we obtain

$$\begin{aligned} A_{1,\alpha}^{(0)} &= \beta^\alpha a_0 w_1^{(0)}(\tau) = \beta^\alpha K_1(\tau), \quad A_{2,\alpha}^{(0)} = \beta^\alpha a_0 w_2^{(0)}(\tau) = \beta^\alpha K_2(\tau), \\ B_1(u) &= B_2(u) = 0, \quad D_\mu = a_1 F(\mu, 1) = a_1 c_\mu = a_1 \beta^\mu = -\beta^{\mu+1}, \end{aligned}$$

where $K_1(\tau)$ and $K_2(\tau)$ were given in the example 1, and finally

$$\begin{aligned}\tilde{Y}_1(t, \tau) = & \frac{1}{2}(e^{-\tau\alpha_1} + e^{-\tau\alpha_2}) \sum_{\alpha=0}^{\infty} \beta^\alpha Y_1(t - \alpha\theta) + \frac{1}{2}(e^{-\tau\alpha_1} - e^{-\tau\alpha_2}) \sum_{\alpha=0}^{\infty} \beta^\alpha Y_2(t - \alpha\theta) \\ & - \sum_{\mu=0}^{\infty} \beta^{\mu+1} Y_1(t + \tau - \theta - \mu\theta).\end{aligned}\quad (5.4)$$

b) If $\tau = \theta$, ($S = N = 1$), then replacing τ by θ , $\tilde{Y} - 1(t, \theta)$ can be obtained from (5.4).

c) If $\tau = \theta$, using the formulae (3.9) and (3.13) – (3.15) we obtain:

$$\begin{aligned}B_1(u) = B_2(u) &= 0, \\ 2A_{1,\alpha}^{(0)} &= 2\beta^\alpha(a_0 K_1(\tau - \theta)) = \beta^\alpha(e^{-\tau\alpha_1} + e^{-\tau\alpha_2} - \beta e^{-(\tau-\theta)\alpha_1} - \beta e^{-(\tau-\theta)\alpha_2}), \\ 2A_{2,\alpha}^{(0)} &= 2\beta^\alpha(a_0 K_2(\tau - \theta)) = \beta^\alpha(e^{-\tau\alpha_1} - e^{-\tau\alpha_2} - \beta e^{-(\tau-\theta)\alpha_1} - \beta e^{-(\tau-\theta)\alpha_2}),\end{aligned}$$

and finally

$$\begin{aligned}2\tilde{Y}_1(t, \tau) = & (e^{\tau\alpha_1} + e^{-\tau\alpha_2} - \beta e^{\tau\alpha_1} - \beta e^{-\tau\alpha_2}) \sum_{\alpha=0}^{\infty} \beta^\alpha Y_1(t - \alpha\theta) + \\ & + (e^{\tau\alpha_1} - e^{-\tau\alpha_2} - \beta e^{-(\tau-\theta)\alpha_1} - \beta e^{-(\tau-\theta)\alpha_2}) \sum_{\alpha=0}^{\infty} \beta^\alpha Y_2(t - \alpha\theta)\end{aligned}\quad (5.5)$$

Example 3. Let $Y(s)$ be a process with the spectral density matrix (5.1) and $X(s) - \beta X(s - 1) = Y(s)$, $|\beta| < 1$. Then $X(s) = AR_Y(1, \theta)$, $N = 1$, $a_0 = 1$, $a_1 = -\beta$ and using the formulae (4.4) – (4.9) we obtain:

$$\begin{aligned}2B_{1,\alpha}^{(0)} &= 2a_\alpha \sum_{\nu=0}^S \beta^\nu K_1(\tau - \nu\theta) = a_\alpha \sum_{\nu=0}^S \beta^\nu (e^{-(\tau-\nu\theta)\alpha_1} + e^{-(\tau-\nu\theta)\alpha_2}) \\ 2B_{2,\alpha}^{(0)} &= 2a_\alpha \sum_{\nu=0}^S \beta^\nu K_2(\tau - \nu\theta) = a_\alpha \sum_{\nu=0}^S \beta^\nu (e^{-(\tau-\nu\theta)\alpha_1} + e^{-(\tau-\nu\theta)\alpha_2})\end{aligned}$$

$$\begin{aligned}D_\mu &= a_0 F(\mu, 0) + a_1 F(\mu, 1) = a_0 c_{S+1-\mu} + a_1 c_{S+\mu} = \beta^{S+1+\mu} - \beta \beta^{S+\mu} = 0, \quad \mu \geq 1 \\ D_0 &= a_0 c_{S+1} = \beta^{S+1} \quad \text{and}\end{aligned}$$

$$\begin{aligned}\tilde{X}_1(t, \tau) = & \frac{1}{2} \sum_{\nu=0}^S \beta^\nu \{e^{-(\tau-\nu\theta)\alpha_1} + e^{-(\tau-\nu\theta)\alpha_2}\} (X_1(t) - \beta X_1(t - \theta)) + \\ & + \frac{1}{2} \sum_{\nu=0}^S \beta^\nu \{e^{-(\tau-\nu\theta)\alpha_1} + e^{-(\tau-\nu\theta)\alpha_2}\} (X_2(t) - \beta X_2(t - \theta)) + \\ & + \beta^{S+1} X_1(t + \tau - (S+1)\theta).\end{aligned}\quad (5.6)$$

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