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A FIXED POINT THEOREM IN A REFLEXIVE BANACH SPACE

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In [1] the following theorem is proved:

THEOREM A Let B a reflexive Banach space, K a nonempty bounded closed and convex subset of B and $T: K \to K$ a mapping satisfying the following conditions:

$$\begin{aligned} \|Tx - Ty\| &\leq \max\{\|x - Tx\|, \|y - Ty\|, (\|x - ty\| + \|y - Tx\|)/3, \\ &(\|x - y\| + \|x - Tx\| + \|yTy\|)/3\}, \ x, y \in K \end{aligned}$$

and

$$\sup_{z \in D} \|z - Tz\| \le \delta(D)/2,$$

where D is any nonempty closed convex subset of K which is mapped into itself by T and $\delta(D) = \sup_{x,y \in D} ||x-y||$ the diameter of D. Then T has a unique fixed point in K.

In the present note we shall prove a theorem which is certain generalization of Theorem A, and its proof is simpler than that of Theorem A. Namely, we have the following:

THEOREM 1. Let B a reflexive Banach space, K a nonempty bounded closed and convex subset of B and $T: K \to K$ a mapping satisfying the following conditions:

(1)
$$\begin{aligned} \|Tx - Ty\| &\leq \max\{\|x - Tx\|, \|y - Ty\|, a\|x - Ty\| + b\|y - Tx\|, \\ (\|x - y\| + \|x - Tx\| + \|y - Ty\|)/3 \\ x, y \in K, \ a \geq 0, b \geq 0, a + b < 1 \end{aligned}$$

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and

(2)
$$\sup_{z \in D} ||z - Tz|| \le r\delta(D), \ 0 \le r = r(D) < 1,$$

where D and $\delta(D)$ have the same meaning as in Theorem A. Then T has a unique fixed point in K.

Proof. Let \mathcal{F} denote the family of all nonempty bounded closed convex subsets of K, which are mapped by T into itself. \mathcal{F} is nonempty since $K \in \mathcal{F}$. If $||F_{\alpha}\rangle$ is any nonincreasing sequence in \mathcal{F} , then by the well known result [2] of Smulian, $F = \bigcap_{\alpha} F_{\alpha}$ is in $\{$. Now, by Zorn's lemma it follows that \mathcal{F} has a minimal element. If C is such a minimal element of \mathcal{F} , we shall prove that C contains only one point, i.e. that T has a fixed point in K. Supposing that C contains more than one element we obtain

(3)
$$\sup_{x,y\in C} \|x-y\| = \delta(C) > 0$$

Since $T(C) \subseteq C$, for any $x, y \in C$ we have, by (1), (2) and (3)

$$||Tx - Ty|| \le \max\{r\delta(C), (a+b)\delta(C), (\delta(C) + 2r\delta(C))/3\}$$

Putting $\overline{r} = \max\{a+b, (1+2r)/3\} < 1$ we have

(4)
$$||Tx - Ty|| \le \{r\delta(C), (\overline{r} < 1) \text{ for each } x, y \in C.$$

If by $\operatorname{co} D$ we denote the convex hull of D, and by $\overline{\operatorname{co}} D$ the closed convex hull of D we have

(5)
$$\overline{\operatorname{co}}T(C) \subseteq \overline{\operatorname{co}}C = \overline{C} = C$$

because $C \in \mathcal{F}, T(C) \subseteq C$ and C is closed and convex. Therefore

(6)
$$T(\overline{\operatorname{co}}T(C)) \subseteq T(C) \subseteq \operatorname{co}T(C) \subseteq \overline{\operatorname{co}}T(C)$$

Since C is a minimal element of \mathcal{F} , by (5) and (6) we have $\overline{\operatorname{co}}T(C) = C$. Let $\overline{x}, \overline{y} \in \operatorname{co}T(C)$. Then we can write

$$\overline{x} = \sum_{i=1}^{n} a_i T x_i, \ a_i \ge 0 \ (i = 1, \dots, n), \quad \sum_{i=1}^{n} a_i = 1, \ x_i \in C$$
$$\overline{y} = \sum_{j=1}^{m} b_j T y_j, \ b_j \ge 0 \ (j = 1, \dots, m), \quad \sum_{j=1}^{n} b_j = 1, \ y_j \in C$$

Now, by (4), and $\sum_{i,j} a_i b_j = 1$

$$\|\overline{x} - \overline{y}\| = \left\|\sum_{i=1}^{n} a_i T x_i - \sum_{j=1}^{m} b_i T y_j\right\| = \left\|\sum_{i,j} a_i b_j T x_i - \sum_{i,j} a_i b_j T y_j\right\| = \left\|\sum_{i,j} a_i b_j (T x_i - T y_j)\right\| \le \sum_{i,j} a_i b_j \overline{r} \delta(C) = \overline{r} \delta(C)$$

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Hence $\|\overline{x} - \overline{y}\| \leq \overline{r}\delta(C)$, for every $\overline{x}, \overline{y} \in \operatorname{co} T(C)$ and therefore

(7)
$$\sup_{\overline{x}, \overline{y} \in \text{co} T(C)} \|\overline{x} - \overline{y}\| \le \overline{r} \delta(C)$$

Now, $\delta(C) = \sup_{x,y \in C} ||x - y||$ and $\overline{\operatorname{co}}T(C) = C$ implies

$$\delta(C) = \sup_{\overline{x}, \overline{y} \in \operatorname{co} T(C)} \|x - y\|, \text{ and by (7) we obtain } \delta \leq \overline{r}\delta.$$

where $\overline{r} < 1$ and $\delta > 0$. This contradiction proves that C contains only one point, i.e. that T has a fixed point in K. Now we shall complete the proof demonstrating that the fixed point of T is unique. Let x_0 and y_0 be two fixed points of T.

Then by (1) we have

$$\begin{aligned} \|x_0 - y_0\| &= \|Tx_0 - Ty_0\| \le \max\{\|x_0 - Tx_0\|, \|y_0 - Ty_0\|, \\ & a\|x_0 - Ty_0\| + b\|y_0 - Tx_0\|, \\ & (\|x_0 - y_0\| + \|x_0 - Tx_0\| + \|y_0 - Ty_0\|)/3\} \\ &= \max\{0, 0, (a+b)\|x_0 - y_0\|, \|x_0 - y_0\|/3\} \end{aligned}$$

From this inequality immediately follows $x_0 = y_0$.

REFERENCES

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