SUBSPACES OF RECURRENT FINSLER SPACES

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1. Introduction. In the Finsler space F_n the metric function is $L(x, \dot{x})$. Let us define the m vector fields $B_a^\alpha(x, \dot{x})$ and $n - m$ vector fields $N_k^\alpha(x, \dot{x})$

$$\alpha, \beta, \gamma, \delta, \varepsilon, \varkappa, \cdots = 1, 2, \ldots, n$$
$$a, b, c, d, e, f, \ \cdots = 1, 2, \ldots, m$$
$$k, l, m, n, p, q, \cdots = m + 1, \ldots, n$$

in such a way that these vector fields are linearly independent at each $(x, \dot{x})$ and satisfy the relations

$$(1.1) \qquad g_{\alpha\beta} = B_a^\alpha N_k^\beta = 0$$

for each $a = 1, 2, \ldots, m \quad k = m + 1, \ldots n$. Let us define

$$(1.2) \qquad g_{ab} = g_{\alpha\beta} B_{ab}^{\alpha\beta}$$

$$(1.3) \qquad g_{kl} = g_{\alpha\beta} N_k^\alpha N_l^\beta$$

$$(1.4) \qquad B_\beta^b = g^{ab} g_{\alpha\beta} B_a^\alpha$$

$$(1.5) \qquad N_\alpha^k = g^{km} g_{\alpha\beta} N_m^\beta$$

$g_{\alpha\beta}$, B_a^α and N_k^α have zero degree of homogenity in $\dot{x}$, g^{ab} and g^{km} are inverse matrices of g_{ab} and g_{km}, respectively. From (1.3) and (1.5) we have

$$(1.6) \qquad N_\alpha^k N_p^\alpha = g^{kl} g_{\alpha\beta} N_l^\beta N_p^\alpha = g^{kl} g_{lp} = \delta_p^k.$$

As usually

$$(1.7) \qquad \delta_\beta^\alpha = B_a^\alpha B_\beta^a + N_k^\alpha N_\beta^k$$

Let us denote the absolute differential which corresponds to the motion from $(x, \dot{x})$ to $(x + dx, \dot{x} + d\dot{x})$ by D. Then we may write

$$(1.8) \qquad dx^\alpha = B_a^\alpha du^a + N_k^\alpha dv^k,$$

$$\dot{x}^\alpha = B_a^\alpha \dot{u}^a + N_k^\alpha N_k^\alpha \dot{v}^k.$$

If $\xi^\alpha(x, \dot{x})$ is a vector field in F_n, then $\xi^\alpha = B_a^\alpha + N_k^\alpha \xi^k$. The induced differentials $\overline{D}\xi^a$, $\overline{D}\xi^k$ are defined by $B_\alpha^a D\xi^\alpha = \overline{D}\xi^a$, $N_\alpha^k D\xi^\alpha = \overline{D}\xi^k$ and

$$(1.9) \qquad D\xi^\alpha = B_a^\alpha \overline{D}\xi^a + N_k^\alpha \overline{D}\xi^k$$

We shall use the notation $l^\alpha = L^{-1}(x, \dot{x})\dot{x}^\alpha = L^{-1}(B_a^\alpha \dot{u}^a + N_k^\alpha \dot{v}^k) = B_a^\alpha + N_k^\alpha l^k$ where $l^a = L^{-1}\dot{u}^a$, $l^k = L^{-1}\dot{v}^k$.

From (1.8) we have

$$(1.10) \qquad Dl^\alpha = B_a^\alpha \overline{D}l^a + N_k^\alpha \overline{D}l^k$$

2. The connection coefficients for the recurrent Finsler space. We shall suppose that the metric tensor is determined by

$$(2.1) \qquad g_{\alpha\beta}(x, \dot{x}) = \dot{\partial}_\alpha \dot{\partial}_\beta L^2(x, \dot{x})/2$$

and that space F_n is recurrent, i.e.

$$(2.2) \qquad Dg_{\alpha\beta} = K(x, \dot{x})g_{\alpha\beta}$$

As

$$Dg_{\alpha\beta} = g_{\alpha\beta|\gamma}dx^\gamma + g_{\alpha\beta}|_\gamma Dl^\gamma$$

(2.2) is valid if

$$(2.3) \qquad g_{\alpha\beta|\gamma} = \lambda_\gamma(x, \dot{x})g_{\alpha\beta}$$

$$(2.4) \qquad g_{\alpha\beta}|_\gamma = \mu_\gamma(x, \dot{x})g_{\alpha\beta}$$

$$(2.5) \qquad K(x, \dot{x}) = \lambda_\gamma(x, \dot{x})dx^\gamma + \mu_\gamma(x, \dot{x})Dl^\gamma.$$

The absolute differential of $g_{\alpha\beta}$ is

$$(2.6) \qquad Dg_{\alpha\beta} - dg_{\alpha\beta} - (\Gamma_{\alpha\gamma}^{*\delta} + \Gamma_{\beta\gamma}^{*\delta}dx^\gamma) - (A_{\alpha\gamma}^\delta g_{\delta\beta} + A_{\beta\gamma}^\delta g_{\alpha\delta})Dl^\gamma.$$

We shall determine connection coefficients under condition (2.2) and

$$(2.7) \qquad \Gamma_{\alpha\gamma}^{*\delta} = \Gamma_{\gamma\alpha}^{*\delta}$$

$$(2.8) \qquad A_{\alpha\gamma}^\delta = A_{\gamma\alpha}^\delta$$

which have a well known geometric interpretation. Substituting

$$(2.9) \quad dg_{\alpha\beta} = \partial_\gamma g_{\alpha\beta}dx^\gamma + \dot{\partial}_\gamma g_{\alpha\beta}d\dot{x}^\gamma =$$

$$= \partial_\gamma g_{\alpha\beta}dx^\gamma + \dot{\partial}_\gamma g_{\alpha\beta}(L_D l^\gamma + l^\gamma DL - \Gamma_\delta^{*\gamma}dx^\delta - LA_{0\delta}^\gamma Dl^\delta)$$

into (2.6) and using (2.3), (2.4) we get

(2.10) $\qquad g_{\alpha\beta|\delta} = \partial_\gamma g_{\alpha\beta} - \dot\partial_\delta g_{\alpha\beta} \Gamma_\gamma^{*\delta} - \Gamma_{\alpha\gamma}^{*\delta} g_{\delta\beta} - \Gamma_{\beta\gamma}^{*\delta} g_{\alpha\delta} = \lambda_\gamma g_{\alpha\beta}$

(2.11) $\qquad g_{\alpha\beta|\gamma} = L\dot\partial_\delta g_{\alpha\beta}(\delta_\gamma^\delta - A_{0\gamma}^\delta) - A_{\alpha\gamma}^\delta g_{\delta\beta} - A_{\beta\gamma}^\delta g_{\alpha\delta} = \mu_\gamma g_{\alpha\beta}.$

Using (2.7) and (2.10) from the expression for $g_{\alpha\beta|\gamma} + g_{\beta\gamma|\alpha} - g_{\gamma\alpha|\beta}$ we obtain

(2.12) $\quad 2\Gamma_{\alpha\beta\gamma}^* = (\partial_\gamma g_{\alpha\beta} + \partial_\alpha g_{\beta\gamma} - \partial_\beta g_{\alpha\gamma}) - -(\dot\partial_\delta g_{\alpha\beta}\Gamma_\gamma^{*\delta} +$
$$\qquad\qquad + \dot\partial_\delta g_{\beta\gamma}\Gamma_\alpha^{*\delta} - \dot\partial_\delta g_{\alpha\gamma}\Gamma_\beta^{*\delta}) - (\lambda_\gamma g_{\alpha\beta} + \lambda_\alpha g_{\beta\gamma} - \lambda_\beta g_{\alpha\gamma})$$

In the same way using (2.11) and (2.8) we have

(2.13) $\quad 2A_{\alpha\beta\gamma} = L(\dot\partial_\alpha g_{\beta\gamma} + \dot\partial_\gamma g_{\alpha\beta} - \dot\partial_\beta g_{\alpha\gamma}) - L(\dot\partial_\delta g_{\alpha\beta} A_{0\alpha}^\delta +$
$$\qquad\qquad + \dot\partial_\delta g_{\alpha\beta} A_{0\gamma}^\delta - \dot\partial_\delta g_{\alpha\gamma} A_{0\beta}^\delta) - (\mu_\alpha g_{\beta\gamma} + \mu_\gamma g_{\alpha\beta} - \mu_\beta g_{\alpha\gamma})$$

On the right-hand side of (2.12) are the undetermined quantities

$$\Gamma_\alpha^{*\delta} = L\Gamma_{0\alpha}^{*\delta}.$$

Multiplying (2.12) by l^α we get

(2.14) $\ 2\Gamma_{0\beta\gamma}^* = (\partial_\gamma g_{\alpha\beta} + \partial_\alpha g_{\beta\gamma} - \partial_\beta g_{\alpha\gamma})l^\alpha - L\dot\partial_\delta g_{\beta\gamma}\Gamma_{00}^{*\delta} - (\lambda_\gamma l_\beta + \lambda_0 g_{\beta\gamma} - \lambda_\beta l_\gamma)$

Multiplying (2.14) by l^γ we obtain

(2.15) $\qquad 2\Gamma_{0\beta 0}^* = (\partial_\gamma g_{\alpha\beta} + \partial_\alpha g_{\beta\gamma} - \partial_\beta g_{\alpha\gamma})l^\alpha l^\gamma - (2\lambda_0 l_\beta + \lambda_\beta)$

and now $\Gamma_{\alpha\beta\gamma}^*$ is determined by (2.12), (2.14) and (2.15).

In the same way from (2.13) we obtain

(2.16) $\qquad 2A_{0\beta\gamma}^= - L\dot\partial_\delta g_{\beta\gamma} A_{00}^\delta - (\mu_\alpha l^\alpha g_{\beta\gamma} - \mu_\gamma l^\beta - \mu_\beta l_\gamma)$

(2.17) $\qquad\qquad\qquad 2A_{0\beta 0} = -(2\mu_0 l_\beta - \mu_\beta).$

3. The induced connection for the subspace of the recurrent space.
Using the method of O. Varga we define DB_a^α and DN_k^α as vectors in F_n which may be decomposed at each lineelement $(x, \dot x)$ in the direction of the vectors B_a^α and N_k^α

(3.1) $\qquad\qquad\qquad DB_a^\alpha = \overline{w}_a^d(d)B_d^\alpha + \overline{w}_a^m(d)N_m^\alpha$

where

(3.2) $\qquad\qquad \overline{w}_a^d(a) = \overline{\Gamma}_{ab}^{*d}du^b + \overline{\Gamma}_{ak}^{*d}dv^k + \overline{A}_{ab}^d\overline{D}l^b + \overline{A}_{ak}^d\overline{D}l^k,$

(3.3) $\qquad\qquad \overline{w}_a^m(a) = \overline{\Gamma}_{ab}^{*m}du^b + \overline{\Gamma}_{ak}^{*m}dv^k + \overline{A}_{ab}^m\overline{D}l^b + \overline{A}_{ak}^m\overline{D}l^k,$

and
$$(3.4) \qquad DN_k^\alpha = \overline{w}_k^d(d)B_d^\alpha + \overline{w}_k^m(d)N_m^\alpha,$$
where
$$(3.5) \qquad \overline{w}_k^d(d) = \overline{\Gamma}_{kb}^{*d}du^b + \overline{\Gamma}_{kn}^{*d}dv^n + \overline{A}_{kb}^d\overline{D}l^b + \overline{A}_{kn}^d\overline{D}l^n$$
$$(3.6) \qquad \overline{w}_k^m(d) = \overline{\Gamma}_{kb}^{*m}du^b + \overline{\Gamma}_{kn}^{*m}dv^n + \overline{A}_{kb}^m\overline{D}l^b + \overline{A}_{kn}^m\overline{D}l^n.$$

From (1.1) and (2.2) we have
$$D(g_{\alpha\beta}B_a^\alpha N_k^\beta) = g_{\alpha\beta}(DB_a^\alpha)N_k^b eta + g_{\alpha\beta}B_a^\alpha DN_k^\beta = 0$$
from which
$$g_{mk}(\overline{\Gamma}_{ab}^{*m}du^b + \overline{\Gamma}_{an}^{*m}dv^n + \overline{A}_{ab}^m\overline{D}l^b + \overline{A}_{an}^m\overline{D}l^n) =$$
$$-g_{ad}(\overline{\Gamma}_{kb}^{*d}du^b + \overline{\Gamma}_{kn}^{*d}dv^n + \overline{A}_{kb}^d\overline{D}l^b + \overline{A}_{kn}^d\overline{D}l^n)$$
follows.

From the relation above we obtain
$$(3.7) \qquad \begin{array}{ll} a) & \overline{\Gamma}_{akb}^* = -\overline{\Gamma}_{kab}^* \\[6pt] b) & \overline{\Gamma}_{akn}^* = -\overline{\Gamma}_{kan}^* \\[6pt] c) & \overline{A}_{akb} = -\overline{A}_{kab} \\[6pt] d) & \overline{A}_{akn} = -\overline{A}_{kan} \end{array}$$

i.e.
$$(3.8) \qquad \overline{w}_{ak} = -\overline{w}_{ka} \quad (g_{km}\overline{w}_b^m = -g_{ab}\overline{w}_k^b).$$

After some calculation using (1.8) and (1.16) we obtain
$$(3.9) \qquad \begin{aligned} DB_a^\alpha &= (B_{a|\beta}^\alpha du^b + B_a^\alpha|_\beta \overline{D}l^b)B_b^\beta + \\ &+ (B_{a|\beta}^\alpha dv^k + B_a^\alpha|_\beta \overline{D}l^k)N_k^\beta \end{aligned}$$
where
$$(3.10) \qquad B_{a|\beta}^\alpha = \partial_\beta B_a^\alpha - \dot\partial_\gamma B_a^k \Gamma_\beta^{*\gamma} + \Gamma_{\gamma\beta}^{*\alpha}B_a^\gamma$$
$$(3.11) \qquad B_a^\alpha|_\beta = L\dot\partial_\delta B_a^\alpha(\delta_\beta^\delta - A_{0\beta}^\delta) + A_{0\beta}^\alpha B_a^\delta$$

Comparing (3.9) and (3.1) using (3.2) and (3.3) we obtain
$$(3.12) \qquad \begin{array}{ll} a) & \overline{\Gamma}_{acb}^* = g_{\alpha\delta}B_c^\delta B_b^\beta B_{a|\beta}^\alpha \\[6pt] b) & \overline{\Gamma}_{ack}^* = g_{\alpha\delta}B_c^\delta N_k^\beta B_{a|\beta}^\alpha \\[6pt] c) & \overline{A}_{acb} = g_{\alpha\delta}B_c^\delta B_b^\beta B_a^\alpha|_\beta \\[6pt] d) & \overline{A}_{ack} = g_{\alpha\delta}B_c^\delta N_k^\beta B_a^\alpha|_\beta \\[6pt] e) & \overline{\Gamma}_{anb}^* = g_{\alpha\delta}N_n^\delta B_b^\beta B_{a|\beta}^\alpha \\[6pt] f) & \overline{\Gamma}_{ank}^* = g_{\alpha\delta}N_n^\delta N_k^\beta B_{a|\beta}^\alpha \\[6pt] g) & \overline{A}_{anb} = g_{\alpha\delta}N_n^\delta B_b^\beta B_a^\alpha|_\beta \\[6pt] h) & \overline{A}_{ank} = g_{\alpha\delta}N_n^\delta N_k^\beta B_a^\alpha|_\beta \end{array}$$

On the other hand we have for the vector field N_k^α

$$(3.13) \qquad DN_k^a lpha = (N_{k|\beta}^\alpha du^b + N_k^\alpha|_\beta \overline{D}l^b)B_b^\beta + (N_{k|\beta}^\alpha dv^m + N_k^\alpha|_\beta \overline{D}l^m)N_m^\beta,$$

where

$$(3.14) \qquad N_{k|\beta}^\alpha = \partial_\beta N_k^\alpha - \dot{\partial}_\gamma N_k^\alpha \Gamma_\beta^{*\gamma} + \Gamma_{\gamma\beta}^{*\alpha} N_k^\gamma$$

$$(3.15) \qquad N_k^\alpha|_\beta = L\dot{\partial}_\delta N_k^\alpha (\delta_\beta^\delta - A_{0\beta}^\delta) + A_{\delta\beta}^\alpha N_k^\delta$$

multiplying (3.4) and (3.13) first with $g_{\alpha\delta}B_c^\delta$ and then with $g_{\alpha\delta}N_n^\delta$, and comparing the coefficients of du^b, dv^l and $\overline{D}l^l$ in these relations we get

$$(3.16)$$

a) $\quad \overline{\Gamma}_{kcb}^* = g_{\alpha\delta}B_c^\delta B_b^\beta N_{k|\beta}^\alpha \qquad$ e) $\quad \overline{\Gamma}_{knb}^* = g_{\alpha\delta}N_n^\delta B_b^\beta N_{k|\beta}^\alpha$

b) $\quad \overline{\Gamma}_{kcl}^* = g_{\alpha\delta}B_c^\delta N_l^\beta N_{k|\beta}^\alpha \qquad$ f) $\quad \overline{\Gamma}_{knl}^* = g_{\alpha\delta}N_n^\delta N_l^\beta N_{k|\beta}^\alpha$

c) $\quad \overline{A}_{kcb} = g_{\alpha\delta}B_c^\delta B_b^\beta N_k^\alpha|_\beta \qquad$ g) $\quad \overline{A}_{knb} = g_{\alpha\delta}N_n^\delta B_b^\beta N_k^\alpha|_\beta$

d) $\quad \overline{A}_{kcl} = g_{\alpha\delta}B_c^\delta N_l^\beta N_k^\alpha|_\beta \qquad$ h) $\quad \overline{A}_{knl} = g_{\alpha\delta}N_n^\delta N_l^\beta N_k^\alpha|_\beta$

4. Recurrent subspaces of recurrent Finsler spaces. From (1.1), (1.2) and (1.3) it follows that

$$(4.1) \qquad g_{\alpha\beta} = g_{ab}B_{\alpha\beta}^{ab} + g_{nk}N_\alpha^n N_\beta^k$$

From (4.1) we obtain

$$(4.2) \qquad \begin{aligned} Dg_{\alpha\beta} = {} & dg_{ab}B_{\alpha\beta}^{ab} + g_{ab}DB_\alpha^a B_\beta^b + g_{ab}B_\alpha^a DB_\beta^b + \\ & + dg_{nk}N_\alpha^n N_\beta^k + g_{nk}DN_\alpha^n N_\beta^k + g_{nk}N_\alpha^n DN_\beta^k \end{aligned}$$

Using (1.7) we have

$$(4.3) \qquad 0 = DB_\beta^b B_b^\gamma + B_\beta^b DB_b^\gamma + DN_\beta^n N_n^\gamma + N_\beta^n DN_n^\gamma.$$

On the other hand

$$g_{ab}B_\beta^b = g_{\alpha\varkappa}B_{ab}^{\alpha\chi} B_\beta^b = g_{\alpha\varkappa}B_a^\alpha (\delta_\beta^\chi - N_k^x N_\beta^n) = g_{\alpha\beta}B_a^\alpha.$$

$$(4.5) \qquad g_{kn}N_\beta^k = g_{\alpha\varkappa}N_k^\alpha N_n^\varkappa N_\beta^k = g_{\alpha\varkappa}N_n^\varkappa (\delta_\beta^\alpha - B_a^\alpha B_\beta^a) = g_{\beta\varkappa}N_n^\varkappa.$$

Using (4.3), (4.4) and (4.5) we have

$$(4.6) \qquad \begin{aligned} g_{ab}DB_\alpha^a B_\beta^b + g_{nk}DN_\alpha^n N_\beta^k &= g_{\varkappa\beta}B_a^\varkappa DB_\alpha^a + g_{\varkappa\beta}N_n^\varkappa DN_\alpha^n = \\ &= g_{\varkappa\beta}(B_\alpha^a DB_a^\varkappa + N_\alpha^n DN_n^\varkappa) \end{aligned}$$

$$(4.7) \qquad \begin{aligned} g_{ab}B_\alpha^a DB_\beta^b + g_{nk}N_\alpha^n DN_\beta^k &= g_{\alpha\varkappa}B_b^\varkappa DB_\beta^b + g_{\alpha\varkappa}N_k^\varkappa DN_\beta^k = \\ &= g_{\alpha\varkappa}(B_\beta^b DB_b^\varkappa + N_\beta^k DN_k^\varkappa) \end{aligned}$$

Substituting (4.6) and (4.7) into (4.2) and using (3.1) and (3.4) we get

$$(4.8) \qquad \begin{aligned} Dg_{\alpha\beta} = {} & B_{\alpha\beta}^{ab}dg_{ab} - g_{\beta\varkappa}B_\alpha^a (B_a^\varkappa \overline{w}_a^d + N_k^\varkappa \overline{w}_a^k) - g_{\beta\varkappa}N_\alpha^n (B_d^\varkappa \overline{w}_n^d + N_k^\varkappa \overline{w}_n^k) + \\ & + N_\alpha^n N_\beta^k dg_{nk} - g_{\alpha\varkappa}B_\beta^b (B_d^\varkappa \overline{w}_b^d + N_k^\varkappa \overline{w}_\beta^k) - g_{\alpha\varkappa}N_\beta^k (B_d^\varkappa \overline{w}_k^d + N_n^\varkappa \overline{w}_n^k). \end{aligned}$$

Using the relations

$$\text{(4.9)} \qquad \begin{aligned} &\text{a)} \quad \dot{u}^a = L\overline{D}l^a - \overline{\Gamma}_c^{*a}du^c - \overline{\Gamma}_k^{*a}dv^k + l^a dL \\ &\text{b)} \quad \dot{v}^k = L\overline{D}l^k - \overline{\Gamma}_c^{*k}du^c - \overline{\Gamma}_l^{*k}dv^l + l^k dL \end{aligned}$$

we obtain

$$\text{(4.10)} \quad \begin{aligned} dg_{ab} &= \partial_c g_{ab}du^c + \partial_k g_{ab}dv^k + \dot{\partial}_c g_{ab}(L\overline{D}l^c - \overline{\Gamma}_d^{*c}du^d - \overline{\Gamma}_k^{*c}dv^k + l^c dL) + \\ &+ \dot{\partial}_k g_{ab}(L\overline{D}l^c - \overline{\Gamma}_d^{*k}du^d - \overline{\Gamma}_t^{*k}dv^t + l^k dL) \end{aligned}$$

(4.10) is valid if g_{ab} is replaced by g_{nk}. Using again (4.4) and (4.5) some terms in (4.8) become zero, i.e.

$$\text{(4.11)} \quad \begin{aligned} &- g_{\beta\varkappa}B_\alpha^a N_k^\varkappa \overline{w}_a^k - g_{\alpha\varkappa}N_\beta^k B_d^\varkappa \overline{w}_k^d = \\ &= -g_{tk}B_\alpha^a N_\beta^t \overline{w}_a^k - g_{da}N_\beta^k B_\alpha^a \overline{w}_k^d = B_\alpha^a N_\beta^t(-\overline{w}_{at} - \overline{w}_{ta}) = 0 \end{aligned}$$

because of (3.8). Similarly we get

$$\text{(4.12)} \qquad -g_{\beta\varkappa}N_\alpha^n B_d^\varkappa \overline{w}_n^d - g_{\alpha\varkappa}B_\beta^b N_k^\varkappa \overline{w}_b^k = -N_\alpha^n B_\beta^b(-\overline{w}_{nb} - \overline{w}_{bn}) = 0.$$

AS $g_{ab} = g_{\alpha\beta}B_a^\alpha B_b^\beta$ is homogeneous of degree zero in $\dot{x}$ so $\dot{\partial}_\varepsilon g_{ab}\dot{x}^\varepsilon = 0$ or

$$\text{(4.13)} \qquad \dot{\partial}_\varepsilon g_{ab}(B_c^\varepsilon \dot{u}^c + N_k^\varepsilon \dot{v}^k) = \dot{\partial}_c g_{ab}\dot{u}^c + \dot{\partial}_k g_{ab}\dot{v}^k = 0$$

Substituting (4.10), (4.11) (4.12) and (4.13) into (4.8) using (4.4), (4.5) we obtain

$$\text{(4.14)} \quad \begin{aligned} Dg_{\alpha\beta} &= B_{\alpha\beta}^{ab}(g_{ab\top c}du^c + g_{ab\top k}dv^k + g_{ab\top c}\overline{D}l^c + g_{ab\top k}\overline{D}l^k) + \\ &N_{\alpha\beta}^{nt}(g_{nt\top c}du^c + g_{nt\top k}dv^k + g_{nt\top c}\overline{D}l^c + g_{nt\top k}\overline{D}l^k), \end{aligned}$$

where

$$\text{(4.15)} \quad \begin{aligned} &\text{a)} \ g_{ab\top x} = \partial_x g_{ab} - \dot{\partial}_d g_{ab}\overline{\Gamma}_x^{*d} - \dot{\partial}_m g_{ab}\overline{\Gamma}_x^{*m} - g_{db}\overline{\Gamma}_{ax}^{*d} - g_{ad}\overline{\Gamma}_{ax}^{*d}, \\ &\quad x = c \ \text{ or } \ x = k, \\ &\text{b)} \ g_{mt\top x} = \partial_x g_{nt} - \dot{\partial}_d g_{ab}\overline{\Gamma}_x^{*d} - \dot{\partial}_m g_{nt}\overline{\Gamma}_x^{*m} - g_{mt}\overline{\Gamma}_{nx}^{*m} - g_{nm}\overline{\Gamma}_{tx}^{*m}, \\ &\quad x = c \ \text{ or } \ x = k, \\ &\text{c)} \ g_{a\top bx} = L\dot{\partial}_x g_{ab} - g_{db}A_{ax}^d - g_{ad}A_{bx}^d, \ \ x = c \ \text{ or } \ x = k, \\ &\text{d)} \ g_{nt\top x} = L\dot{\partial}_x g_{nt} - g_{mt}A_{nx}^m - g_{nm}A_{tx}^m, \ \ x = c \ \text{ or } \ x. = k, \end{aligned}$$

Using the relation

$$\text{(4.16)} \qquad Dg_{\alpha\beta} = B_c^\delta g_{\alpha\beta|\delta}du^c + N_k^\delta g_{\alpha\beta|\delta}dv^k + B_c^\delta g_{\alpha\beta}|_\delta Dl^c + N_k^\delta g_{\alpha\beta}|_\delta \overline{D}l^k$$

from (4.14) we obtain

$$\text{(4.17)} \qquad\qquad\qquad B_c^\delta g_{\alpha\beta\top\delta} = B_{\alpha\beta}^{ab}g_{ab\top c} + N_{\alpha\beta}^{nt}g_{nt\top c}$$

$$\text{(4.18)} \qquad\qquad\qquad N_k^\delta g_{\alpha\beta|\delta} = B_{\alpha\beta}^{ab}g_{ab\top k} + N_{\alpha\beta}^{nt}g_{nt\top k}.$$

Multiplying (4.17) by B_γ^c and (4.18) by N_γ^k and adding these relations we get

$$(4.19) \qquad g_{\alpha\beta|\gamma} = B_{\alpha\beta\gamma}^{abc} g_{ab\top c} + B_{\alpha\beta}^{ab} N_\gamma^k g_{ab\top k} + N_{\alpha\beta}^{nt} B_\gamma^c g_{nt\top c} + N_{\alpha\beta\gamma}^{ntk} g_{nt|k}.$$

In the same way we obtain

$$(4.20) \qquad B_c^\delta g_{\alpha\beta}|_\delta = B_{\alpha\beta}^{ab} g_{ab}\top_c + N_{\alpha\beta}^{nt} g_{nt}\top_c$$

$$(4.21) \qquad N_k^\delta g_{\alpha\beta}|_\delta = B_{\alpha\beta}^{ab} g_{ab}\top_k + N_{\alpha\beta}^{nt} g_{nt}\top_k$$

$$(4.22) \qquad g_{\alpha\beta}|_\gamma = B_{\alpha\beta\gamma}^{abc} g_{ab}\top_c + B_{\alpha\beta}^{ab} N_\gamma^k g_{ab}\top_k + N_{\alpha\beta}^{nt} B_\gamma^c g_{nt}\top_c + N_{\alpha\beta\gamma}^{ntk} g_{nt}\top_k.$$

From (4.19) we have

LEMMA 4.1. *If*

$$(4.23) \qquad g_{ab|c} = \lambda_c g_{ab}, \;\; g_{ab|k} = \lambda_k g_{ab}, \;\; g_{nt|c} = \lambda_c g_{nt}, \;\; g_{ab|k} = \lambda_k g_{nt}$$

then $g_{\alpha\beta|\gamma} = \lambda_\gamma g_{\alpha\beta}$ where,

$$(4.24) \qquad \lambda_\gamma = B_\gamma^c \lambda_c + N_\gamma^k \lambda_k.$$

and λ_c, λ_k, λ_γ are covariant vectors.

Proof. Substituting (4.23) and (4.24) into (4.19) we have

$$g_{\alpha\beta|\gamma} = B_{\alpha\beta}^{ab}(B_\gamma^c \lambda_c g_{ab} + N_\gamma^k \lambda_k g_{ab}) + N_{\alpha\beta}^{nt}(B_\gamma^c \lambda_c g_{nt} + N_\gamma^k \lambda_k g_{nt} =$$
$$\lambda_\gamma B_{\alpha\beta}^{ab} g_{ab} + \lambda_\gamma N_{\alpha\beta}^{nt} g_{nt} = \lambda_\gamma g_{\alpha\beta}.$$

In the same way from (4.22) we have

LEMMA 4.2. *If*

$$(4.25) \qquad g_{ab|c} = \mu_c g_{ab}, \;\; g_{ab|k} = \mu_k g_{ab}, \;\; g_{nt|c} = \mu_c g_{nt}, \;\; g_{ab|k} = \mu_k g_{nt}$$

then $g_{\alpha\beta|\gamma} = \mu_\gamma g_{\alpha\beta}$ where,

$$(4.26) \qquad \mu_\gamma = B_\gamma^c \mu_c + N_\gamma^k \mu_k.$$

From (4.19) we have

LEMMA 4.3. *If $g_{\alpha\beta|\gamma} = \lambda_\gamma g_{\alpha\beta}$ then $g_{ab|c} = \lambda_c g_{ab}, \; g_{nt|c} = \lambda_c g_{nt}, \; g_{nt|k} = \lambda_k g_{nt}$ where:*

$$(4.27) \qquad \lambda_c = B_c^\gamma \lambda_\gamma, \;\; \lambda_k = N_k^\gamma \lambda_\gamma$$

Proof. Multiplying (4.19) by $B_d^\gamma \; B_e^\beta \; B_f^\alpha$ and using the relations:

$$(4.28) \qquad N_\alpha^k B_c^\alpha = 0 \quad B_a^\alpha B_\alpha^c = \delta_a^c \quad N_k^\alpha N_\alpha^n = \delta_k^n$$

we get

$$g_{\alpha\beta|\gamma} B_{def}^{\gamma\beta\alpha} = \delta_f^a \delta_e^b \delta_d^c g_{ab|c} \Rightarrow \lambda_\gamma B_d^\gamma g_{\alpha\beta} B_{fe}^{\alpha\beta} = \lambda_\gamma B_d^\gamma g_{je} = g_{fe|d}.$$

Multiplying (4.19) by $B_f^\alpha \ B_e^\beta \ N_n^\gamma$ and using (4.28) we have

$$g_{\alpha\beta|\gamma} B_f^\alpha B_e^\beta N_n \gamma = \delta_{fen}^{abk} g_{ab|k} \Rightarrow \lambda_\gamma g_{\alpha\beta} B_f^\alpha B_e^\beta N_n^\gamma = \lambda_\gamma N_n^\gamma g_{fe} = g_{fe|n}.$$

Multiplying (4.19) by $N_{kl}^{\alpha\beta} \ B_d^\gamma$ and using (4.28) we have

$$g_{\alpha\beta|\gamma} N_{kl}^{\alpha\beta} B_d^\gamma = \delta_k^n \delta_l^t \delta_d^c g_{nt|c} \Rightarrow \lambda_\gamma g_{\alpha\beta} N B_{kl}^{\alpha\beta} B_d^\gamma = \lambda_\gamma B_d^\gamma g_{kl} = g_{kl|d}.$$

Multiplying (4.19) by $N_{plm}^{\alpha\beta\gamma}$ and using (4.28) we obtain

$$g_{\alpha\beta|\gamma} N_{plm}^{\alpha\beta\gamma} = \delta_{plm}^{ntk} g_{nt|k} \Rightarrow \lambda_\gamma g_{\alpha\beta} N plm^{\alpha\beta\gamma} = \lambda_\gamma N_m^\gamma g_{pl} = g_{pl|d}.$$

LEMMA 4.4 *If* $_{\alpha\beta}|_\gamma = \mu_\gamma g_{\alpha\beta}$, *then* $g_{ab}|_c = \mu_c g_{ab}$, $g_{ab}|_k = \mu_k g_{ab}$, $g_{nt}|_c = \mu_c g_{nt}$, $g_{nt}|_k = \mu_k g_{nt}$, *where* $\mu_c = B_c^\gamma \mu_\lambda$, $\mu_k = N_k^\lambda \mu_\lambda$.

The proof is similar to that of Lemma 4.3 if we start from (4.22).

From lemma 4.1 and 4.3 we have

THEOREM 4.1. *The necessary and sufficient conditions for* $g_{\alpha\beta|\gamma} = \lambda_\gamma g_{\alpha\beta}$ *are* (4.23) *and* (4.24)

From lemmas 4.2 and 4.4 we have

THEOREM 4.2. *The necessary and sufficient conditions for* $g_{\alpha\beta|\gamma} = \mu_\gamma g_{\alpha\beta}$ *are* (4.25) *and* (4.26)

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