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# ON SUMS INVOLVING RECIPROCALS OF CERTAIN ARITHMETICAL FUNCTIONS

*Paul Erdös, Aleksandar Ivić*

**1. Introduction and statement of results.** Let as usual $\omega(n)$ and $\Omega(n)$ denote the number of distinct prime factors and the number of total prime factors of $n$ respectively, and let $P(n)$ denote the largest prime factor of $n\geqslant 2$. Our aim is to estimate certain arithmetical sums (some of these estimates were posed an open problems in [4]) which will exhibit the similarities and diferences in behaviour of $\omega(n)$ and $\Omega(n)$. Our first results are contained in the following theorems.

Theorem 1. *For $x\geqslant x_0$ and some constants $0<C_2<C_1$ we have*

$$x\exp(-C_1(\log x\cdot\log\log x)^{1/2})\leqslant\sum_{2\leqslant n\leqslant x}n^{-1/\omega(n)}\leqslant x\exp(-C_2(\log x\cdot\log\log x)^{1/2}). \tag{1.1}$$

Theorem 2. *For $x\geqslant x_0$ and some constants $0<C_3<C_4$ we have*

$$x\exp(-C_3(\log x)^{1/2})\leqslant\sum_{2\leqslant n\leqslant x}n^{-1/\Omega(n)}\leqslant x\exp(-C_4(\log x)^{1/2}). \tag{1.2}$$

Here $\exp y=e^y$, and these theorems show that the sum appearing in (1.1) is of a lower order of magnitude than the sum in (1.2). Further we conjecture that for some $C,\ D>0$

$$\sum_{2\leqslant n\leqslant x}n^{-1/\omega(n)}=x\exp(-(C+o(1))(\log x\cdot\log\log x)^{1/2}), \tag{1.3}$$

$$\sum_{2\leqslant n\leqslant x}n^{-1/\Omega(n)}=x\exp(-D+o(1))\log x)^{1/2}). \tag{1.4}$$

It was shown in [7] (for a still sharper result see [8]) that

$$\sum_{2\leqslant n\leqslant x}1/P(n)=x\exp\left(-\left(\sqrt{2}+o(1)\right)(\log x\cdot\log\log x)^{1/2}\right), \tag{1.5}$$

and it seems interesting to investigate how the sum in (1.5) is affected when $1/P(n)$ is replaced by $\Omega(n)/P(n)$ or $\omega(n)/P(n)$. We are able to do this in the former case, and if as usual $f \ll g$ (which is the same as $f = O(g)$) means that $|f| \leqslant Cg$ for some absolute $C > 0$, then we have

Theorem 3.

$$(1.6) \qquad (\log x/\log\log x)^{1/2} \sum_{2 \leqslant n \leqslant x} 1/P(n) \ll \sum_{2 \leqslant n \leqslant x} \Omega(n)/P(n) \ll$$

$$\ll (\log x \log\log x)^{1/2} \sum_{2 \leqslant n \leqslant x} 1/P(n).$$

For some related problems with $\omega(n)$ we refer the reader to [6], and we remark that the upper and lower bound in (1.6) differ only by a factor of $\log\log x$. A classical result of G.H. Hardy and S. Ramanujan (see [9]) states that $\omega(n)$ and $\Omega(n)$ both have average and normal order $\log\log n$, but Theorem 3 shows that replacing $1/P(n)$ by $\Omega(n)/P(n)$ changes the sum by a factor of $(\log x)^{1/2+o(1)}$, which is much larger than the normal order of $\Omega(n)$. It may be also remarked that Theorem 3 remains valid if $P(n)$ is replaced by

$$(1.7) \qquad \beta(n) = \sum_{p \mid n} p, \text{ or } B(n) = \sum_{p^a \| n} ap.$$

Here and later $p$ denotes primes; $a \mid b$ means that $a$ divides $b$ and $p^k \| m$ means $p^k \mid m$ and $p^{k+1} \nmid m$. The functions $\beta(n)$ and $B(n)$ are additive and they were investigated in [1], [4] Ch. 6, [5], [7] and [8]. Though the difference in asymptotic behaviour of sums with $1/P(n)$, $1/\beta(n)$ and $1/B(n)$ was discussed in [5], in Theorem 3 this is irrelevant in view of (1.5) and

$$(1.8) \qquad P(n) \leqslant \beta(n) \leqslant B(n) \ll P(n) \log n,$$

so that following the same proof we obtain (1.6) with $P(n)$ replaced by $\beta(n)$ or $B(n)$.

To formulate our last result suppose that $\omega(n) \geqslant r$, where $r \geqslant 2$ is a fixed integer, and further suppose that $P_1(n) > P_2(n) > \cdots > P_r(n)$ are the $r$ largest prime factors of $n$ written in decreasing order. In this notation $P(n) = P_1(n)$, and it may be mentioned that questions involving $P_r(n)$ were discussed extensively in [1] and [2]. The asymptotic behaviour of the summatory function of $1/P_1(n)$ is given by (1.5), and it is interesting to note that a different type of asymptotic formula holds if we consider the summatory function of $1/P_r(n)$, $r \geqslant 2$. This is shown by

Theorem 4. *There is a constant $B_2 > 0$ such that*

$$(1.9) \qquad \sum_{n \leqslant x}{}' 1/P_2(n) = B_2 x/\log x + O(x(\log\log x)^2/\log^2 x),$$

*and if $i \geqslant 3$ is a fixed integer there is a constant $B_i > 0$ such that*

$$(1.10) \qquad \sum_{n \leqslant x}{}' 1/P_i(n) = (B_i + o(1))\, x(\log\log x)^{i-2}/\log x.$$

In (1.9) $\sum'$ denotes summation over those $n$ for which $\omega(n)\geqslant 2$, while in (1.10) $\sum'$ denotes summation over those $n$ for which $\omega(n)\geqslant i$.

**2. Proof of Theorem 1 and Theorem 2.** We begin by proving the lower bound in (1.1). Let $A_k$ be as on p. 152 of [4] defined by

$$A_k=\{n:(n\leqslant x)\wedge(\mu^2(n)=1)\wedge(P(n)\leqslant x^{1/k})\}, \tag{2.1}$$

where $\mu$ is the Möbius function and $k(\ll\log x/\log\log x)$ is a large integer which will be suitably chosen. Since $\mu^2(n)=1$ means that $n$ is squarefree, i.e. that $n$ is a product of different primes, and since by the prime number theorem there are at least $U=3kx^{1/k}/(4\log x)$ primes not exceeding $x^{1/k}$, we have

$$\sum_{n\in A_k,\ \omega(n)=k} 1\geqslant\binom{U}{k}=\frac{U(U-1)\cdots(U-k+1)}{k!}\geqslant\left(\frac{2}{3}U\right)^k/k!,$$

provided that

$$U-k+1\geqslant 2U/3. \tag{2.2}$$

Noting that for $k\geqslant 6$ we have $(k/2)^k\geqslant k!$ it follows that

$$\sum_{n\in A_k,\ \omega(n)=k} 1\gg x\log^{-k}x. \tag{2.3}$$

Therefore we obtain

$$\sum_{2\leqslant n\leqslant x} n^{-1/\omega(n)}\geqslant\sum_{n\in A_k,\ \omega(n)=k} x^{-1/k}\gg x\exp(-k\log\log x-(\log x)/k),$$

and the choice $k=k(x)=[(\log x/\log\log x)^{1/2}]$ gives the desired bound in (1.1), since (2.2) is easily seen to hold in this case.

We pass now to the proof of the upper bound in (1.1), abbreviating $\log\log x=\log_2 x$ and letting $C$ denote (possibly different) positive, absolute constants. We shall use a classical inequality of G.H. Hardy and S. Ramanujan ([9], p. 265):

$$\sum_{n\leqslant x,\ \omega(n)=k} 1<Ex\log^{-1}x(\log_2 x+F)^k/k!, \tag{2.4}$$

where $E,\ F>0$ are absolute constants. We have

$$\sum_{2\leqslant n\leqslant x} n^{-1/\omega(n)}=S_1+S_2+O(x^{1/2}), \tag{2.5}$$

where first trivially

$$S_2=\sum_{x^{1/2}<n\leqslant x,\ \omega(n)<(\log n/\log_2 n)^{1/2}} n^{-1/\omega(n)}\leqslant \tag{2.6}$$

$$\sum_{x^{1/2}<n\leqslant x}\exp(-(\log n\log_2 n)^{1/2})\ll x\exp(-C(\log x\log_2 x)^{1/2}).$$

With $k=[C(\log x/\log_2 x)^{1/2}]$ we have from (2.4)

$$\sum_{n\leq x,\, \omega(n)\geq k} 1 \ll x \sum_{j\geq k} (C\log_2 x)^j/j! \ll x\exp\left(+Ck\log_3 x-\frac{k}{2}\log k\right) \ll \tag{2.7}$$

$$x\exp(-C(\log x\log_2 x)^{1/2}),$$

where we used $\log k! > (k\log k)/2$. This gives then

$$S_1 = \sum_{x^{1/2}<n\leq x,\, \omega(n)\geq(\log n/\log_2 n)^{1/2}} n^{-1/\omega(n)} \ll \tag{2.8}$$

$$\sum_{n\leq x,\, \omega(n)\geq C(\log x/\log_2 x)^{1/2}} 1 \ll x\exp(-C(\log x\log_2 x)^{1/2}),$$

and the upper bound in Theorem 1 follows from (2.5), (2.6) and (2.8).

To prove the lower bound of Theorem 2 we consider integers of the form $r=2^t m$, where $m\geq 1$ is an integer and $t=[\log^{1/2} x]$. We have $\Omega(r)\geq t$, and so for $r\leq x$ we have $r^{-1/\Omega(r)}\geq x^{-1/t}$ which gives

$$\sum_{2\leq n\leq x} n^{-1/\Omega(n)} \geq \sum_{r\leq x} x^{-1/t} \gg x^{1-1/t}\,2^{-t} \gg x\exp(-C\log^{1/2} x),$$

as asserted. For the upper bound in (1.2) we need the estimate

$$\sum_{n\leq x,\, \Omega(n)\geq k} 1 \ll x\,2^{-k/4}, \quad \text{if } k\geq(\log\log x)^2, \tag{2.9}$$

where the $\ll$-constant is uniform in $k$ and $x$. This follows from (2.4) and

Lemma 1. *Uniformly in $k$ we have*

$$\sum_{n\leq x,\, \Omega(n)-\omega(n)\geq k} 1 \ll x\,2^{-k/2}. \tag{2.10}$$

To prove (2.10) it suffices to assume that $k$ is an integer and that $\Omega(n)-\omega(n)=k$. Every $n>1$ may be written uniquely as $n=qs$, $(q,s)=1$, where $q$ is squarefree and $s$ is squarefull (meaning $p\mid s$ implies always $p^2\mid s$). But then we have $\Omega(n)-\omega(n)=\Omega(s)-\omega(s)$, and from

$$s\geq 2^{\Omega(s)}\geq 2^{\Omega(s)-\omega(s)}=2^k$$

we infer that the left-hand side of (2.10) is bounded by

$$\sum_{s\leq x,\, s\geq 2^k}\ \sum_{q\leq x/s} 1 \ll x\sum_{s\geq 2^k} s^{-1} \ll x\,2^{-k/2}.$$

Here we used the estimate $\sum_{s\geq y} s^{-1} \ll y^{-1/2}$, which follows by partial summation from the lementary estimate $\sum_{s\leq y} 1 \leq y^{1/2}$.

To obtain (2.9) we write simply

$$\sum_{n\leqslant x,\,\Omega(n)\geqslant k} 1 \leqslant \sum_{n\leqslant x,\,\omega(n)\geqslant k/2} 1 + \sum_{n\leqslant x,\,\Omega(n)-\omega(n)\geqslant k/2} 1$$

and use (2.4) (similarly as (2.7) is established) for the first sum on the right-hand side above and (2.10) for the second sum.

To finish the proof of theorem 2 we use (2.9) with $k=\log^{1/2}x$. Then we have

$$\sum_{2\leqslant n\leqslant x} n^{-1/\Omega(n)} = S' + S'' + O(x^{1/2}), \tag{2.11}$$

where

$$S' = \sum_{x^{1/2}<n\leqslant x,\,\Omega(n)\leqslant k} n^{-1/\Omega(n)} \ll \sum_{n\leqslant x} x^{-1/2k} \ll x\exp(-C\log^{1/2}x), \tag{2.12}$$

$$S'' = \sum_{x^{1/2}<n\leqslant x,\,\Omega(n)>k} n^{-1/\Omega(n)} \leqslant \sum_{n\leqslant x,\,\Omega(n)\geqslant k} 1 \ll x\exp(-C\log^{1/2}x), \tag{2.13}$$

so that the upper bound in (1.2) follows from (2.11) — (2.13).

**3. Proof of Theorem 3.** We begin the proof of Theorem 3 and Theorem 4 by showing that in (1.6) we may restrict the range of summation to these $n\leqslant x$ for which

$$P(n)\leqslant \exp(4(\log x\log_2 x)^{1/2}) = H(x), \tag{3.1}$$

since trivially

$$\sum_{n\leqslant x,\,P(n)>H(x)} \Omega(n)/P(n) \ll x\log x\cdot\exp(-4(\log x\log_2 x)^{1/2}).$$

and (1.5) gives the true order of magnitude of $\sum_{2\leqslant n\leqslant x}\Omega(n)/P(n)$ as

$$x\exp\left(-\left(\sqrt{2}+o(1)\right)(\log x\log_2 x)^{1/2}\right),$$

since $1\leqslant\Omega(n)\ll\log x$ for $2\leqslant n\leqslant x$. Threefore for $n$ satisfying (3.1) we have

$$n\leqslant (P(n))^{\Omega(n)}\leqslant \exp(4\,\Omega(n)(\log x\log_2 x)^{1/2}). \tag{3.2}$$

Since the contribution of $n\leqslant x^{1/2}$ is trivially $\ll x^{1/2}$ we may suppose also that $x^{1/2}<n\leqslant x$, and so (3.2) gives

$$\Omega(n)\gg(\log x/\log\log x)^{1/2} \tag{3.3}$$

for $n$ satisfying (3.1) and $x^{1/2}<n\leqslant x$, producing at once the lower bound in Theorem 3.

For the upper bound in Theorem 3 we use (2.9) with $k=10(\log x\log_2 x)^{1/2}$. If $\sum'$ denotes summation over those $n\leqslant x$ for which (3.1) holds, then

$$\sum{}' \Omega(n)/P(n)=\sum_{\Omega(n)<k}{}' \Omega(n)/P(n)+\sum_{\Omega(n)\geqslant k}{}' \Omega(n)/P(n)\ll \tag{3.4}$$

$$(\log x\log x_2)^{1/2}\sum{}' 1/P(n)+x\exp(-2(\log x\log_2 x)^{1/2})\ll(\log x\log_2 x)^{1/2}\sum_{2\leqslant n\leqslant x} 1/P(n),$$

where we used (1.5). This establishes the upper bound in (1.6).

**4. Proof of Theorem 4.** We shall give a detailed proof of (1.9), and indicate only the proof of the somewhat weaker general estimate (1.10). We begin by making first several simplifying assumptions.

a) By an estimate of N.G. de Bruijn [3] we have

$$\sum_{n\leqslant x,\, P_1(n)\leqslant\exp(\log x/\log_2 x)} 1\ll x\log^{-2} x,$$

so we may assume that

$$p=P_1(n)>\exp\log x/\log_2 x). \tag{4.1}$$

b) We may assume that $P_1^2(n)$ does not divide $n$, for otherwise

$$\sum_{n\leqslant x,\, p^2\mid n,\, p>\exp(\log x/\log_2 x)} 1\ll x\sum_{p>\exp(\log x/\log_2 x)} p^{-2}\ll x\log^{-2} x.$$

c) Suppose that $q=P_2(n)$ and $q^b\,\|\,n$. Then we may assume $q^b\leqslant\log^2 x$, since

$$\sum_{x\leqslant x,\, q>\log^2 x} 1/q\ll x\log^{-2} x,$$

$$\sum_{mq^b\leqslant x,\, q^b>\log^2 x,\, b\geqslant 2} 1/q\ll x\sum_{q^b>\log^2 x,\, b\geqslant 2} q^{-b-1}\ll x\sum_{b\leqslant\log x/\log 2} 1/(b\log^2 x)$$

$$\ll x\log_2 x/\log^2 x.$$

d) We may assume that if $n=pq^b r$ with $P_1(r)<q$ and $r\geqslant 1$, then $r\leqslant\exp(20(\log_2 x)^2)=F$. To see this note that if $s^a$ divides $r$ ($a\geqslant 2$, $s$ prime) then similarly as in *c)* we may assume that $s^a\leqslant\log^4 x$. Further we may assume $\omega(r)\leqslant 5\log_2 x$, since otherwise the number of $n\leqslant x$ for which $\omega(n)>5\log_2 x$ is by [6] $O(x\log^{-2})$ and then

$$r\leqslant(\log^4 x)^{\omega(r)}\leqslant\exp(20(\log_2 x)^2)=F,$$

because if $s\,\|\,r$, then by *c)* we may assume $s<q\leqslant\log^2 x$.

We proceed now with the proof of (1.9), noting first that by *d)* and the prime number theorem there are

$$xq^{-b}r^{-1}\log^{-1} x(1+O(\log_2^2 x/\log x))$$

prime numbers in the interval $(\exp(\log x/\log_2 x),\ xq^{-b}r^{-1}]$. Therefore we have

$$\sum_{n\leq x}{}' 1/P_2(n) = O(x\log_2^2 x/\log^2 x) + \sum_{q^b\leq\log^2 x} q^{-1} \sum_{P(r)<q,\, r\leq F}\ \sum_{\exp(\log x/\log_2 x)<p\leq xq^{-b}r^{-1}} 1$$

$$= O(x\log_2^2 x/\log^2 x) + x\log^{-1} x \sum_{q^b\leq\log^2 x} q^{-b-1} \sum_{P(r)<q} 1/r =$$

$$= B_2 x/\log x + O(x\log_2^2 x/\log^2 x).$$

Here we used the fact that the sum $S=\sum_q q^{-b-1} \sum_{P(r)<q} r^{-1}$ is bounded. To see this note that

$$\sum_{P(r)<q} r^{-1} = \sum_j \sum_{p_1<\cdots<p_j<q,\, a_1\geq 1,\ldots,\, a_j\geq 1} p_1^{-a_1}\cdots p_j^{-a_j} \ll$$

$$\sum_j (\log\log q+3)^j/j! \leq \exp(\log(2\log q)) = 2\log q,$$

and this implies that $S$ is bounded. Furthermore the contribution of $q^b>\log^2 x$ and $r>F$ is $O(x\log_2^2\log^2 x)$, since

$$\sum_{pq^b r\leq x,\, P(r)<q,\, r>F} q^{-1} \ll x\sum_{r>F} r^{-1} \sum_{q^b>P(r)} q^{-b-1} \ll x\sum_{r>F}(rP(r))^{-1}\log x \ll$$

$$x\exp(-0.5(\log F\log_2 F)^{1/2}) \ll x\log^{-2} x,$$

where we used partial summation and (1.5) to estimate $\sum_{r>F}(rP(r))^{-1}$. This completes the proof of (1.9), where with $q$ prime we have

$$B_2 = \sum_q f(q)/(q^2-q);\ f(2)=1,\ f(q) = \sum_{F(r)<q} r^{-1} \text{ if } q>2.$$

The proof of (1.10) is similar to the proof of (1.9), but technically more difficult. The main term comes from ($q$, $r$, $s_j$ will be all primes in what follows)

$$x\log^{-1} x \sum_{q^b\leq\log^2 x} q^{-1-b} \sum_{P(r)\leq q,\, r\leq F} r^{-1} \sum_{\substack{q<s_1<\cdots<s_{i-2}<p \\ s_1^{a_1}\leq\log^4 x,\ldots,\, s_{i-2}^{a_{i-2}}\leq\log^4 x}} s_1^{-a_1}\cdots s_{i-2}^{-a_{i-2}} = \cdots =$$

$$x\log^{-1} x \sum_{q^b\leq\log^2 x} q^{-1-b} \sum_{P(r)<q,\, r\leq F} r^{-1} \sum_{s_{i-2}<x} (c_i+O(1))\, s_{i-2}^{-1}(\log\log s_{i-2})^{i-3} =$$

$$= (B_i+o(1))\, x(\log_2 x)^{i-2}/\log x.$$

Here ... denotes that summation is being carried over $s_1, \ldots, s_{i-3}$ with the use of

$$\sum_{p\leqslant x} p^{-1}(\log_2 p)^k = \int_{3/2}^{x} t^{-1}(\log_2 t)^k \, d\pi(t) = (1+o(1)) \int_{3/2}^{x} t^{-1}\log^{-1} t\,(\log_2 t)^k \, dt$$

$$= ((k+1)^{-1} + o(1))(\log_2 x)^{k+1}$$

which follows from the prime number theorem $\pi(x) = \sum_{p\leqslant x} 1 = \int_{3/2}^{x} \frac{dt}{\log t} +$ $+ O(x\log^{-k} x)$ $k>0$ is any fixed number). The $o$-term in (1.10) could be replaced by a $O$-term at the cost of some tehnical complications in the proof.

## REFERENCES


[1] K. Alladi, P. Erdös, *On an arithmetic function*, Pacific J. Math. **71** (1977), 275—294.

[2] K. Alladi, P. Erdös, *On the asymptotic behaviour of large prime factors of an integer*, Pacific J. Math. **82**(1979), 295—315.

[3] N.G. de Bruijn, *On the number of positive integers $\leqslant x$ and free of prime factors $> y$*, Nederl. Akad. Wet. Proc. **54**(1951), 50—60 and ibid. **28**(1966), 239—247.

[4] J. -M. De Koninck, A. Ivić, *Topics in arithmetical functions*, Notas de Matematica 72, Amsterdam, 1980.

[5] P. Erdös, A. Ivić, *Estimates for sums involving the largest prime factor of an integer and certain related additive functions*, Studia Scien. Math. Hungarica, **15** (1980), 183—199.

[6] P. Erdös, J. -L. Nicolas, *Sur la fonction »Nombre de facteurs premiers de n«*, Séminaire Delange—Pisot—Poitou 1978/79, No. 32. 19pp.

[7] A. Ivić, *Sum of reciprocals of the largest prime factor of at integer*, Arch. Math. **36** (1980), 57—61.

[8] A. Ivić, C. Pomerance, *Estimates of certain sums involving the largest prime factor of an integer*, to appear in Coll. Math. Soc. János Bolyai 34, Topics in classical number theory, North--Holland, Amsterdam.

[9] S. Ramanujan, *Collected Papers*, Chelsea, New York, 1962.



Mathematical Institute
Hungarian Academy of Sciences
Reáltanoda utca 13—15
H-1053 Budapest V, Hungary

Katedra Matematike
RGF-a Univerziteta u Beogradu
Djušina 7, 11000 Beograd
Jugoslavija


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