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COMMON FIXED POINT THEOREMS FOR COMMUTING MAPS ON A METRIC SPACE

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Abstract. Results generalizing fixed point theorems of various authors and generalizing and unifying fixed point theorems of Jungck and Boyd-Wong are established.

Introduction. The purpose of the present note (which is entirely conceptual) is to present three common fixed point theorems. The first one follows J. Matkowski's ideas ([6]). The second one generalizes a fixed point theorem by the author ([2]). The third one, on the contrary, generalizes and unifies fixed point theorems by G. Jungck ([5]) and Boyd-Wong ([1]), following K. M. Das and K. Viswanatha Naik's ideas ([3]).

Everywhere in this paper (E, d) is a complete metric space. Furthermore, in 1 and 2 g will be a self-map on E.

Recently Matkowski proved the following

Theorem 1. *Let* $\alpha:[0, +\infty)^5 \to [0, +\infty)$ *and let* $\psi(t) = \alpha(t, t, t, 2t, 2t)$ *for* $t \geqslant 0$. *Suppose that*

1) *for every* $x \in E$ *there exists a positive integer* $n = n(x)$ *such that for all* $y \in E$, $d(g^n x, g^n y) \leqslant \alpha(d(x, g^n x), d(x, g^n y), d(x, y), d(y, g^n x), d(y, g^n y))$,

2) α *is non-decreasing with respect to each variable*,

3) $\lim\limits_n \psi^n(t) = 0, \qquad t > 0$,

4) $\lim\limits_{t \to +\infty} (t - \psi(t)) = +\infty$.

Then g *has a unique fixed point* $u \in E$ *and for each* $x \in E$, $\lim\limits_n g^n(x) = u$.

Now, let f be a self-map on E that commutes with g such that $\overline{f(E)} = f(E)$ and $g(E) \subset f(E)$. If we replace 1) and 4) above by the condition

$1)_1$ *for every* $x \in E$ *there exists a positive integer* $n = n(f(x))$ *such that*

$$d(g^n(f(x)), g^n(f(y))) \leqslant \alpha(d(f(x), g^n(f(x))), d(f(x), g^n(f(y))), d(f(x), f(y)), d(f(y), g^n(f(x))), d(f(y), g^n(f(y)))),$$

$4)_1$ *there exists* $x \in E$ *such that* $\max\limits_{i \leqslant n(f(x))} d(f(x), g^i(f(x))) < \lim\limits'_{t \to +\infty} (t - \psi(t))$,

then we obtain a common fixed-point for f and g.

In **2**, if $\psi:[0, +\infty)\to[0, +\infty)$ is non-decreasing continuos on the right, $\psi(t)<t$ for each $t>0$, f is a continuous self-map on E that commutes with g such that $g(E)\subset f(E)$, and if, moreover, we suppose that

a) *for every* $x\in E$ *there exists a positive integer* $n=n(f(x))$ *such that* $d(g^n(f(x)), g^n(f(y)))\leqslant\psi(d(f(x), f(y)))$ *for all* $y\in E$,

b) *there exists an* $x\in E$ *such that, assuming* $\beta=\max\limits_{i\leqslant n(f(x))} d(f(x), g^i(f(x)))$, *there exists a* $\delta>0$ *such that* $\beta<\delta-\psi(\delta)$,

then we obtain a common fixed-point theorem, of which a previous result of the author ([2]) is a particular case.

In **3**, if $\psi:[0, +\infty)\to[0, +\infty)$ is upper semicontinuous from the right and satisfies $\psi(t)<t$ for every $t>0$, f is a self-map on E such that f^m, where m is any fixed positive integer, is continuous, $g: f^{m-1}(E)\to E$ is a map that commutes with f and, moreover, we suppose that

c) $g(f^{m-1}(E))\subset f^m(E)$

d) $d(g(x), g(y))\leqslant\psi(d(f(x), f(y))$ *for every* $x, y\in f^{m-1}(E)$

then we obtain a common fixed point theorem of which Jungck's theorem is a particular case.

The following points are worth emphasizing:

— Theorem 2 and 3 can be naturally extended to a Hausdorff uniform space the uniformity of which is generated by a non empty family of pseudo--metric on it;

— To the best of the author's knowledge, no trace can be found in the literature of relations between g and ψ of the kind indicated by conditions $4)_1$ and b);

— condition $4)_1$, implies b). However, condition b) cannot be used instead of $4)_1$, at least not for the kind of proof presented below.

Finally, we shall use the following notations: N = the set of all positive integers, R_+ = the set of all nonnegative real numbers, $R_+^*=R_+-\{0\}$, $\lim'=\lim\inf$.

1. We begin this section by

Lemma 1. *Suppose that* $\psi: R_+\to R_+$ *is non-decreasing. If* ψ^n, $n=0, 1\ldots$ *denotes the n-th iterate of* ψ *and if for every* $t>0$, $\lim\limits_n \psi^n(t)=t$, *then* $\psi(t)<t$ *holds.*

Lemma 2. *Let* $\alpha: R_+^5\to R_+$, $f: E\to E$ *and let* $\psi(t)=\alpha(t, t, t, 2t, 2t)$ *for* $t\geqslant 0$. *Suppose that*

i) α *is non-decreasing with respect to each variable,*

ii) *for every* $x\in E$, *there exists a positive integer* $n=n(f(x))$ *such that* $d(g^n(f(x)), g^n(f(y)))\leqslant\alpha(d(f(x), g^n(f(x))), d(f(x), g^n(f(y))), d(f(x), f(y)), d(g^n(f(x)), f(y)), d(g^n(f(y)), f(y)))$ *for all* $y\in E$,

iii) *$x \in E$ exists such that* $\max_{i \leqslant n(f(x))} d(f(x), g^i(f(x))) < \lim'_{t \to +\infty} (t - \psi(t))$. *Then* $\sup_n d(f(x), g^n(f(x))) < +\infty$ *holds for every $x \in E$ that satisfy condition* iii).

Proof. Let x be a point in E that satisfies iii) and put $y = f(x)$, $n = n(f(x))$ and $\beta = \max_{i \leqslant n(f(x))} d(f(x), g^i(f(x)))$. By iii) there exists $\delta \in R_+^*$, $\beta < \delta$ such that $\beta < t - \psi(t)$ for all $t \geqslant \delta$. Let $r \in N$, $0 \leqslant r < n$ and put $d_k = d(y, g^{kn+r} y)$, $k = 0, 1, \ldots$. Let $j = \min\{i \in N \mid \delta \leqslant d_i\}$. Evidently, $d_i < \delta$ for $i < j$.

Hence, by the triangle inequality

$$d(g^n y, g^{(j-1)n+r} y) \leqslant d(g^n y, y) + d(y, g^{(j-1)n+r} y) \leqslant \beta + d_{j-1} < 2 d_j$$

$$d(g^{jn+r} y, g^{(j-1)n+r} y) \leqslant d(g^{jn+r} y, y) + d(y, g^{(j-1)n+r} y) \leqslant d_j + d_{j-1} < 2 d_j.$$

Now, using i) and iii), one gets

$$d_j = d(y, g^{jn+r} y) \leqslant d(y, g^n y) + d(g^n y, g^{jn+r} y) \leqslant$$

$$\leqslant \beta + \alpha(d(y, g^n y), d(y, g^{jn+r} y), d(y, g^{(j-1)n+r} y), d(g^n y, g^{(j-1)n+r} y),$$

$$d(g^{jn+r} y, g^{(j-1)n+r} y)) \leqslant \beta + \alpha(d_j, d_j, d_j, 2 d_j, 2 d_j) = \beta + \psi(d_j),$$

i.e. $d_j - \psi(d_j) < \beta$, which together with $\delta \leqslant d_j$ contradicts the choice of δ. Therefore $d_i < \delta$ for $i = 0, 1, \ldots$ and, consequently, $\sup_n d(y, g^n y) < +\infty$.

Theorem 2. *Let $\alpha: R_+^5 \to R_+$, lef f be a self-map on E that commutes with g, such that $\overline{f(E)} = f(E)$ and $g(E) \subset f(E)$, and let $\psi(t) = \alpha(t, t, t, 2t, 2t)$. Assume that conditions* $1)_1$ — 2) — 3) — $4)_1$ *are fulfilled. Then f and g have a unique common fixed point.*

Proof. Let x be a point in E that satisfies condition iii) and put $y = f(x)$. We define a sequence of points $(y_n)_{n \in N}$ as follows. Let $y_0 = y$, $m_0 = n(y_0)$ and $y_n = g^{m_{n-1}} y_{n-1}$, $m_n = n(y_n)$ for each $n \in N$, $1 \leqslant n$. Evidently, $(y_n)_{n \in N}$. is a subsequence of $(g^n y)_{n \in N}$. We show that $(y_n)_{n \in N}$ is a Cauchy sequence. It suffices to show that for a given $\varepsilon > 0$, $d(y_{n+1}, y_{n+k+1}) < \varepsilon$ for all $k \in N$, when n is large enough. For this purpose, let $n \in N$ be fixed, $d_i = d(y_n, g^i y_n)$ and $m(k) = m_{n+1} + m_{n+2} + \cdots + m_{n+k}$. Then

$$d(g^{m_n} y_n, g^{m(k)} y_n) \leqslant d(g^{m_n} y_n, y_n) + d(y_n, g^{m(k)} y_n) = d_{m_n} + d_{m(k)}$$

$$d(g^{m_n}(g^{m(k)} y_n), g^{m(k)} y_n) \leqslant d(g^{m_n}(g^{m(k)} y_n), y_n) + d(y_n, g^{m(k)} y_n) = d_{m_n + m(k)} + d_{m(k)}.$$

Hence, if t_1 denotes a number chosen among $m(k)$, m_n, $m(k) + m_n$ such that $d_{t_1} = \max(d_{m(k)}, d_{m_n}, d_{m(k)+m_n})$, we have

$$d(y_{n+1}, y_{n+k+1}) = d(g^{m_n} y_n, g^{m_{n+k}} y_{n+k}) = d(g^{m_n} y_n, g^{m_n}(g^{m(k)} y_n)) \leqslant$$

$$\leqslant \alpha(d(y_n, g^{m_n} y_n), d(y_n, g^{m_n + m(k)} y_n), d(y_n, g^{m(k)} y_n), d(g^{m_n} y_n, g^{m(k)} y_n),$$

$$d(g^{m_n + m(k)} y_n, g^{m(k)} y_n)) \leqslant \alpha(d_{t_1}, d_{t_1}, d_{t_1}, 2 d_{t_1}, 2 d_{t_1}) = \psi(d_{t_1}).$$

Repeating this procedure, we can find positive integers $t_j, j=1, \ldots, n$ such that $d(y_{n-j+1}, g^{t_j} y_{n-j+1}) \leqslant \psi(d(y_{n-j}, g^{t_j+1} y_{n-j}))$.

Hence, since ψ is nondecreasing, we obtain

$$d(y_{n+1}, y_{n+k+1}) \leqslant \psi^{n+1}(d(y, g^{t_{n+1}} y)) \leqslant \psi^{n+1}(\sup_n d(y, g^n y))$$

with $\sup_n d(y, g^n y) < +\infty \cdot (y_n)_{n \in N}$ is a Cauchy sequence as it derives from 3). Now, since (E, d) is a complete metric space, the Cauchy sequence defined above converges to a point, say $u \in E$. By an argument similar to that used above, one can easily show that $\lim_n d(g^{n(u)} y_n, y_n) = 0$. Now, let $\varepsilon = d(g^{n(u)} u, u) > 0$; there exists an $n_0 \in N$ such that

$$d(u, y_n) < (\varepsilon - \psi(\varepsilon))/4, \qquad d(g^{n(u)} y_n, y_n) < (\varepsilon - \psi(\varepsilon))/4$$

for all $n \in N$, $n_0 \leqslant n$. Keeping in mind that $u \in \overline{f(E)} = f(E)$, it follows that

$$\begin{aligned}\varepsilon = d(g^{n(u)} u, u) &\leqslant d(g^{n(u)} u, g^{n(u)} y_n) + d(g^{n(u)} y_n, y_n) + d(y_n, u) \leqslant \\ &\leqslant \alpha(d(u, g^{n(u)} u), d(u, g^{n(u)} y_n), d(u, y_n), d(g^{n(u)} u, y_n), d(g^{n(u)} y_n, y_n)) + \\ &+ (\varepsilon - \psi(\varepsilon))/2 \leqslant \alpha(\varepsilon, \varepsilon, \varepsilon, 2\varepsilon, 2\varepsilon) + (\varepsilon - \psi(\varepsilon))/2 = \psi(\varepsilon) + (\varepsilon - \psi(\varepsilon))/2 < \varepsilon\end{aligned}$$

which is a contradiction. Consequently, $g^{n(u)}(u) = u$.

Suppose that there is a $v \in E$ such that $g^{n(u)}(v) = v$. Since

$$\begin{aligned}d(u, v) = d(g^{n(u)} u, g^{n(u)} v) \leqslant \alpha(&d(u, g^{n(u)} u), d(u, g^{n(u)} v), d(u, v), d(g^{n(u)} u, v), \\ &d(v, g^{n(u)} v)) \leqslant \psi(d(u, v))\end{aligned}$$

one has $d(u, v) = 0$, i.e. $u = v$. Then, since $g(u) = g^{n(u)}(g(u))$, u is a fixed point of g. Finally, $f(u) = f(g(u)) = g(f(u))$; hence $f(u) = u$, i.e. u is a fixed point of f. This completes the proof.

Remark 1. If in Theorem 2 we take $f = I_E$ (the identity map on E), we get Theorem 2 of [2].

Remark 2. If in Theorem 2 we take $f = I_E$, $n(x) = 1$ for all $x \in E$, we get Theorem 1 of Husain-Sehgal ([4]).

2. Next we shall consider the proof of the following theorem.

Theorem 3. *Let $\psi: R_+ \to R_+$ be non decreasing, continuous on the right, $\psi(t) < t$ for each $t > 0$ and let f be a continuous self-map on E that commutes with g and such that $g(E) \subset f(E)$. Assume that conditions* a) — b) *are fulfilled. Then f and g have a unique common fixed point.*

Proof. Let u and v be such that $f(u) = g(u) = u$ and $f(v) = g(v) = v$. Since

$$\begin{aligned}d(u, v) = d(g(u), g(v)) &= d(g^{n(f(u))}(f(u)), g^{n(f(u))}(f(v))) \leqslant \\ &\leqslant \psi(d(f(u), f(v))) = \psi(d(u, v))\end{aligned}$$

we have $d(u, v) = 0$, i.e., $u = v$, so that uniqueness is obtained.

Now, let x be a point in E that satisfies b); by an argument similar to that used for Theorem 2, one can easily show that the sequence $(g^n(f(x))_{n\in N}$ contains a Cauchy subsequence $(y_n)_{n\in N}$. Let $u\in E$ be such that $u=\lim_n y_n$. By the continuity of f we have $f(u)=\lim_n f(y_n)$. Since for every $n\in N$

$$d(y_n, f(y_n))=d(g^{m_n-1}y_{n-1}, g^{m_n-1}(f(y_{n-1})))\leqslant$$
$$\leqslant\psi(d(y_{n-1}, f(y_{n-1})))\leqslant\cdots\leqslant\psi^n(d(y, f(y)))$$

by the continuity of ψ, we have $\lim_n d(y_n, f(y))=0$. It follows that $d(u, f(u))=0$, i.e., $u=f(u)$. Hence u is a fixed point of f. Now we note that

$$d(g^{n(u)}u, g^{n(u)}y_n)\leqslant\psi(d(u, y_n))\leqslant d(u, y_n)$$

so that $\lim_n g^{n(u)}y_n=g^{n(u)}(u)$. Hence $d(g^{n(u)}u, u)=\lim_n d(g^{n(u)}y_n, y_n)$. Since

$$d(g^{n(u)}y_n, y_n)\leqslant\psi(d(g^{n(u)}y_{n-1}, y_{n-1}))\leqslant\cdots\leqslant\psi^n(d(y, g^{n(u)}y))\leqslant$$
$$\leqslant\psi^n(\sup_n d(y, g^n y))^{(1)}$$

we have $\lim_n d(g^{n(u)}y_n, y_n)=0$, so that $d(g^{n(u)}u, u)=0$, i.e., $g^{n(u)}u=u$. Then by condition a), u is the unique fixed point of $g^{n(u)}$ and, hence, it is also the unique fixed point of g, since

$$g^{n(u)}(g(u))=g(g^{n(u)}u)=g(u).$$

This completes the proof.

Remark 3. If in Theorem 3 we take $f=i_E$, we get the Corollary of [2].

Remark 4. If in Theorem 3 f is a nonexpansive self-map on E that commutes with g, and if we replace a) and b) by the conditions

a)$_1$ *for every* $x\in E$ *there exists a positive integer* $n=n(x)$ *such that* $d(g^n(x), g^n(y))\leqslant\psi(d(f(x), f(y)))$ *for all* $y\in E$,

b)$_1$ *there exists an* $x\in E$ *and a* $\delta>0$ *such that* $\beta<\delta-\psi(\delta)$, *where*

$$\beta=\max_{i\leqslant n(x)} d(x, g^i(x)),$$

then the conclusion of Theorem 3 *holds again.*

3. Next we consider the proof of the following theorem.

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(1) By an argument similar to that used for Lemma 2, one can easily show that condition b) implies $\sup_n d(y, g^n y)<+\infty$ with $y=f(x)$, where $x\in E$ is the x considered in b).

Theorem 4. *Let $\psi: R_+ \to R_+$ be upper semicontinuous from the right and let it satisfy $\psi(t) < t$ for all $t > 0$. Let f be a self-map on E such that f^m, where m is any fixed positive integer, is continuous and let $g: f^{m-1}(E) \to E$ commute with f. Assume that conditions* e) — d) *are fulfilled. Then f and g have a unique common fixed point.*

Proof. Starting with an arbitrary point x_0 in $f^{m-1}(E)$ and appealing to condition c), we construct a sequence $(x_n)_{n \in N}$ of points in $f^{m-1}(E)$ such that $f(x_{n+1}) = g(x_n)$.

Let $y_n = f(x_{n+1}) = g(x_n)$ for every $n \in N$. Note that $f(y_n) = f(g(x_n)) = g(f(x_n)) = g(y_{n-1})$. Let $z_n = f(y_n)$ for every $n \in N$. By an argument similar to that used in [1] by Boyd-Wong (cf. Theorem 1), one can easily show that the sequence $(z_n)_{n \in N}$ is a Cauchy sequence of points in E. Let $z \in E$ be such that $z = \lim_n z_n$. By the continuity of f^m, $(f^m(z_n))_{n \in N}$ converges to $f^m(z)$. Moreover, $g(f^{m-1}(z_n)) = g(f^{m-1}(f^2(x_{n+1}))) = f^m(g(f(x_{n+1}))) = f^m(f(g(x_{n+1}))) = f^m(z_{n+1})$ implies that $(g(f^{m-1}(z_n)))_{n \in N}$ converges to $f^m(z)$. Furthermore,

$$d(f^m(z_{n+1}), g(f^{m-1}(z))) = d(g(f^{m-1}(z_n)), g(f^{m-1}(z))) \leqslant \psi(d(f^m(z_n), f^m(z))) \leqslant$$
$$\leqslant d(f^m(z_n), f^m(z))$$

so that $g(f^{m-1}(z)) = \lim_n f^m(z_{n+1}) = f(z)$. Finally,

$$d(g(g(f^{m-1}(z))), g(f^{m-1}(z))) \leqslant \psi\, d(f(g(f^{m-1}(z))), f^m(z))) =$$
$$= \psi(d(g(g(f^{m-1}(z))), g(f^{m-1}(z))))$$

yields $g(g(f^{m-1}(z))) = g(f^{m-1}(z))$; $g(f^{m-1}(z))$ can be easily seen to be a fixed point of f too. Hence f and g have a common fixed point. The uniqueness follows once again from condititon b).

Remark 5. If in Theorem 4 we take $f = i_E$, we get Theorem 1 of Boyd-Wong.

Remark 6. If in Theorem 4 we take $\psi(t) = k \cdot t$ (where $0 < k < t$) and $m = 1$ we get Jungck's fixed point theorem. We also remark that in Theorem 4 f is not necessarily continuous, whereas in Jungck's theorem it is.

Remark 7. Contractive inequalities, usually supposed in common fixed point theorems for two self-maps f and g on a complete metric space (E, d) (still with the conditions $f \circ g = g \circ f$ and $g(E) \subset f(E)$), are of the following form

$$(*) \qquad d(g(x), g(y)) \leqslant \cdots \qquad \text{(cf. Theorem 4)}$$

whereas the inequalities supposed in Theorems 2 and 3 are of the following form

$$(**) \qquad d(g(f(x)), g(f(y))) \leqslant \cdots$$

Then it is worth noting the link between the two types of theorems. Common fixed point theorems, with the inequality (**), are direct consequences of theorems with the inequality (*). Generally the converse is not true. But if f is one-to-one, the two types of theorems are equivalent.

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