PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 32 (46) 1982 pp. 33—36

ON A RANDOM FUNCTION DEFINED ON A PSEUDO BOOLEAN ALGEBRA

Jelena Bulatović

Denote by $\mathcal{A}=(A, \cup, \cap, \rightarrow, -)$ any pseudo Boolean algebra [2], by $\wedge$ and $\vee$ the minimal and maximal element in A, respectively, and by $\geqslant$ the ordering relation in A. Suppose that the function p, defined on A, has the following properties (see [1]):

(I) $p(a)\geqslant 0$ for any $a\in A$,

(II) $p(\vee)=1$,

(III) if $a_1, a_2, \cdots \in A$ are such that $a_i\cap a_j=\wedge$ for $i\neq j$, and

$$\bigcup_{i=1}^{\infty} a_i \text{ exists, then } p\Big(\bigcup_{i=1}^{\infty} a_i\Big)=\sum_{i=1}^{\infty} p(a_i),$$

(IV) if $a\leqslant b$, $a, b\in A$, then $p(a)\leqslant p(b)$;

such a function p will be called probability on $\mathcal{A}$. It is obvious that, if $\mathcal{A}$ it a Boolean algebra, then this is the classical definition of probability.

In this paper we shall define one mapping with values in A and show that, if $\mathcal{A}$ is a Boolean algebra, then that mapping is isomorphic to the inverse function of some random variable defined in the classical way. Later we shall define different types of convergence of a sequence of so defined mappings and, also, investigate relationships between these types of convergence.

In order to do what we just promised we need some preliminaries. Denote by R the real line, by $\mathcal{B}(R)$ the Boolean σ-algebra of Borel sets from R, and by i the ordinary non-trivial interior operation on R; then $\mathcal{T}(R)=(\mathcal{B}(R), \cup, \cap, C, i)$ is the topological Boolean algebra [2].

Any operation H from the set $i(R)$ of all open sets from R into A will be called a homomorhism from $i(R)$ into $\mathcal{A}$ if the following conditions are satisfied:

(i) $H(\varnothing)=\wedge$, $H(R)=\vee$,

(ii) if $A, B\in i(R)$ are such that $A\cap B=\varnothing$, then $H(A)\cap H(B)=\wedge$,

(iii) if $A_1, A_2, \cdots \in i(R)$ are such that $A_i\cap A_j=\varnothing$ for $i\neq j$, and

$$\bigcup_{i=1}^{\infty} H(A_i) \text{ exists, then } H\Big(\bigcup_{i=1}^{\infty} A_i\Big)=\sum_{i=1}^{\infty} H(A_i).$$

If $\mathcal{A}$ is a pseudo Boolean σ-algebra, then a homomorphism H becomes a σ-homomorphism.

When i is the ordinary non-trivial interior operation on $\mathcal{B}(R)$, and H is any homomorphism from $i(R)$ into $\mathcal{A}$, then the mapping $X = H \cdot i : B \in \mathcal{B}(R) \rightarrow X(B) = H(i(B)) \in A$ will be called the random function on $\mathcal{A}$.

In the next theorem we shall prove that, in the case $\mathcal{A}$ is a Boolean σ-algebra, the random function X is isomorphic to the inverse function of one random variable, defined in the classical way.

Theorem. *Let* $(\Omega, \mathcal{F})$ *be a Boolean* σ*-algebra and* h *a* σ*-homomorphism from* $\mathcal{F}$ *onto a Boolean* σ*-algebra* $\mathcal{A}$. *If* X *is a random function on* $\mathcal{A}$, *then there exists a real-valued* $\mathcal{F}$*-measurable function* Y *on* Ω (*i.e. a random variable in the classical sense*), *such that*

$$X(B) = h(Y^{-1}(B)) \tag{1}$$

for any $B \in \mathcal{B}(R)$.

Proof. Let $r_1, r_2, \ldots$ be a sequence of all rational numbers; put $B_n = (-\infty; r_n)$ and $A_n = X(B_n) = H(i(B_n))$, $n = 1, 2, \ldots$. First, we shall construct a sequence $E_1, E_2, \ldots$ of elements from $\mathcal{F}$, corresponding to the sequence $r_1, r_2, \ldots$ and having the following two properties:

(a) $h(E_i) = A_i$ for $i = 1, 2, \ldots$,

(b) if $r_i < r_j$, $i, j = 1, 2, \ldots$, then $E_i \subset E_j$.

The assumption that h is a homomorphism onto A implies the existence of at least one E_1 from $\mathcal{F}$, such that

$$h(E_1) = A_1.$$

For the same reason, there exists $E \in \mathcal{F}$, such that $h(E) = A_2$; it is clear that such an E is not necessarily unique. If we want to find $E_2 \in \mathcal{F}$ satisfying also (b) for $i = 1, 2, \ldots$, then we have to compare r_1 and r_2, and for each of two possible cases to define E_2.

If $r_2 < r_1$, then $A_2 \leqslant A_1$ (for H is a homomorphism), which implies that from $A_2 \neq \wedge$ it follows that $E_1 \cap E \neq \varnothing$. Let us show that the equality

$$h(E_1 \cap E) = A_2 \tag{2}$$

is satisfied. Using (ii) we have

$$A_1 \cap h(E \setminus E_1) = \wedge \text{ and } A_2 = h(E) = h(E \setminus E_1) \cup h(E \cap E_1),$$

which, together with the previous equality, implies

$$h(E \setminus E_1) = \wedge,$$

and thus (2) is proved. This permits us to define the set E_2 in the case $r_2 < r_1$, by $E_2 = E_1 \cap E$.

But, if $r_2 > r_1$, then $A_2 \geqslant A_1$ and, again using (ii), we have $h(E_1 \cup E) = A_2 \cup h(E_1 \setminus E)$, which, together with $h(E_1 \setminus E) \leqslant h(E_1) = A_1 \leqslant A_2$, implies

$h(E_1 \setminus E) = \wedge$; that means that the element $E_1 \cup E$ also satisfies the equality $h(E_1 \cup E) = A_2$, which permits us, in the case $r_2 > r_1$, to define the set E_2 by $E_2 = E_1 \cup E$.

Suppose that we already have constructed, for some finite n, the elements $E_1, E_2, \dots, E_n$ from $\mathcal{F}$, satisfying the conditions (a) and (b). Let $E \in \mathcal{F}$ be any element such that the equality $h(E) = A_{n+1}$ holds; then the element E_{n+1} will be defined by

$$E_{n+1} = \begin{cases} \left(\bigcap_{i=1}^{n} E_i\right) \cap E, & \text{if } r_{n+1} < r_i \text{ for all } i = 1, 2, \dots, n, \\ E_m \cup (E \cap E_M), & \text{if } r_m < r_{n+1} < r_M, \\ \left(\bigcup_{i=1}^{n} E_i\right) \cup E, & \text{if } r_{n+1} > r_i \text{ for all } i = 1, 2, \dots, n, \end{cases}$$

where

$$r_m = \max\{r_i : r_i < r_{n+1},\ i = 1, 2, \dots, n\}, \qquad r_M = \min\{r_i : r_i > r_{n+1},\ i = 1, \dots, n\}.$$

It is easy to see that in this way we constructed a sequence $E_1, E_2, \dots$ having the properties (a) and (b).

Let us put $F = \bigcup_{i=1}^{\infty} E_i$, and define the function Y on Ω by

$$Y(\omega) = \begin{cases} 0, & \text{if } \omega \in \Omega \setminus F, \\ \inf\{r_n : \omega \in E_n\}, & \text{if } \omega \in F; \end{cases}$$

this definition is correct, since complete ordering of the set $\{E_i\}$ implies that, for any $\omega \in \Omega$, there exists a positive integer j, such that $\omega \notin E_j$ and $\omega \in E_{j+1}$.

In order to show that the function Y is $\mathcal{F}$-measurable, it is sufficient to show that for any t the set $Y^{-1}((-\infty;\ t))$ belongs to $\mathcal{F}$, which immediately follows from the equality

$$Y^{-1}((-\infty;\ t)) = \begin{cases} \bigcup_{r_i < t} E_i, & t \leq 0, \\ \bigcup_{r_i < t} E_i \cup (\Omega \setminus F), & t > 0. \end{cases} \tag{3}$$

Now, we have only to prove that (1) holds. It is easy to see (since h and H are homomorphisms) that $h\left(\bigcup_{n=1}^{\infty} E_n\right) = \vee$, which implies the equality $h(\Omega \setminus F) = \wedge$; the last equality, together with (2), gives

$$h(Y^{-1}((-\infty;\ t))) = \bigcup_{r_i < t} A_i = X((-\infty;\ t))$$

for any real t. From that equality and from the fact that $\mathcal{B}(R)$ is generated by all sets of the form $(-\infty;\ t)$, $t \in R$, it follows that (1) is satisfied for any $B \in \mathcal{B}(R)$. This proves the theorem.

Corollary. *If all conditions from the above theorem are satisfied, then the σ-field $\mathcal{F}$ can be replaced by some other σ-field $\overline{\mathcal{F}}$ in such a way that h becomes a σ-isomorphism.*

Proof. It is clear that the relation $\sim$, defined on $\mathcal{F}$ by

$$G_1 \sim G_2 \text{ if and only if } h(G_1) = h(G_2),$$

is an equivalence relation on $\mathcal{F}$. The equivalence class generated by $G \in \mathcal{F}$, will be denoted by $|G|$. Let us put $\overline{\mathcal{F}} = \mathcal{F}/\sim$, and suppose that all operations on $\overline{\mathcal{F}}$ are defined obviously: $|G_1| \cup |G_2| = |G_1 \cup G_2|$, $|G_1| \cap |G_2| = |G_1 \cap G_2|$, $\overline{|G|} = |\overline{G}|$. It is easy to see that $\overline{\mathcal{F}}$ is a σ-field and that h is a σ-isomorphism from $\overline{\mathcal{F}}$ onto $\mathcal{A}$. The proof is completed.

It is clear that, when h is an isomorphism, the sets $E_1, E_2, \ldots,$ are uniquely determined, and hence the function Y is also uniquely defined, that is there exists just one random variable (in the classical sense) such that the mappings Y^{-1} and X are isomorphic.

REFERENCES

[1] J. Bulatović, *On probability function on pseudo-Boolean algebra*, Publ. Inst. Math. **24(38)**(1978) 29—38.

[2] H. Rasiowa, R. Sikorski, *The mathematics of metamathematics*, Warszawa 1963.

Matematički Institut
Knez Mihailova 35
11000 Beograd

(Received 21 03 1981)