ON THE FREQUENCY WITH WHICH BOUNDED FUNCTIONS HAVE SPECTRAL GAPS

Richard Bieberich

Some special results were proved in [3] and [9] whose theme seems to be that it is more common for a bounded function on the real line to have spectrum all of R than to have a spectral gap. There are several ways to quantify such a statement.

Let X be a set of objects or natural equivalence classes of objects potentially capable of exhibiting some phenomenon (such as having spectrum all of R), and let $A \subset X$ be the subset of objects which actually exhibit it. A counting argument might show that the cardinality of A is strictly greater than the cardinality of $X—A$. Alternatively, a natural topology or measure might be defined on X with respect to which either A is topologically large, or $X—A$ is topologically small or of measure zero. Such arguments incidentally establish that the phenomenon does occur.

In this paper the fragments of [3] and [9] are put into a proper framework: two simple proofs, one topological and the other a cardinality argument, will be given that most bounded measurable functions have no spectral gaps. The main ingredient in the notion of a spectral gap is analytic continuation of holomorphic functions in half-planes. In many ways it is profitable to view such functions as analogues of power series, so that the arguments given here are naturally motivated by the classical results that most power series do not continue beyond their circles of convergence (see, for example, [2], pp. 91—104). The necessary definitions and facts concerning the spectrum of a bounded function are collected below. These are followed by the results on the frequency with which bounded functions have spectral gaps, and the paper closes with some general remarks.

Let f be in $L^p(R)$ for $1 \leqslant p \leqslant \infty$. The two functions

$$F^+(f, z) = \int_{-\infty}^{0} f(t)e^{-izt}\, dt \quad \text{and} \quad F^-(f, z) = -\int_{0}^{\infty} f(t)e^{-izt}\, dt$$

are holomorphic in the upper and lower half-planes, respectively. This pair of functions is called the Fourier-Carleman transform of f. They are clearly linear in the first argument. The complement of the open set on the real line across which F^+ and F^- continue analytically to each other is called the spectrum of f, and is denoted by $\sigma(f)$. An open interval in the complement of $\sigma(f)$ is

called a spectral gap of f. One motivation for the interest in $\sigma(f)$ is that, when f is both bounded and integrable, $\sigma(f)$ and the support of the ordinary Fourier transform of f are the same [8, pp. 179-180].

(1)   **L e m m a.** *If f is bounded and if there is a positive number β such that*

$$\int_0^\infty |f(t)| e^{\beta t} dt$$

is finite, then $F^-(f, z)$ is holomorphic on $Im\ z < \beta$.

P r o o f. If $z = x + iy$, where $y < \beta$, then

(2)
$$F^-(f, z) = F^-(f, x + iy) = -\int_0^\infty f(t) e^{-i(x+iy)} dt =$$

$$= -\int_0^\infty f(t) e^{\beta t} e^{-i(x+i[y-\beta])t} dt = F^-(\psi, x + i[y - \beta]),$$

where $\psi(t) = f(t)e^{\beta t}$. ψ is integrable on all of R, so $F^-(\psi, x + i[y - \beta])$ is holomorphic when $y - \beta = Im(x + i[y - \beta]) < 0$, or when $Im\ z < \beta$. Equation (2) then says that $F^-(f, z)$ is also holomorphic when $Im\ z < \beta$.

The closure of the trigonometric polynomials

$$\sum_{n=1}^N a_n e^{i\lambda_n x}, \quad \lambda_n \text{ real},$$

with respect to different notions of distance yields different classes of almost periodic functions [1, p. 70 ff.]. The Bohr almost periodic functions are obtained by the uniform closure; thus they are plainly bounded. It is proved in [5] that the spectrum of a Bohr almost periodic function is the closure of its set of Fourier exponents (the word "spectrum" is not used there). Thus any closed subset F of the real line can be the spectrum of a bounded function: if $\{\lambda_n \mid n \in N\}$ is dense in F and if $\sum_{n=1}^\infty |a_n|$ converges, then $\sum_{n=1}^\infty a_n e^{i\lambda_n x}$ is a Bohr almost periodic function whose spectrum is F. Hence the spectrum of a bounded function can be discrete; in particular, this occurs for a periodic function.

Both quantitative statements, Theorems (8) and (10), describing the frequency with which a bounded function has a spectral gap, follow easily from the following uniqueness property of the Fourier-Carleman transform.

(3)   **T h e o r e m.** *Let f be a bounded function on the real line. If there is a positive number β such that*

4)
$$\int_0^\infty |f(t)| e^{\beta t} dt$$

is finite and if f has a spectral gap, then f is zero almost everywhere.

This result was stated, but not proved, in [11, p. 151].

P r o o f. Since (4) is finite, (1) says that $F^-(f, z)$ is holomorphic on $Im\ z < \beta$. But $F^-(f, z)$ continues across f's spectral gap to $F^+(f, z)$, so, by the Identity Theorem for holomorphic functions [7, p. 199],

(5) $$F^-(f, z) = F^+(f, z)$$

on the strip $0 < Im\ z < \beta$.

Let $0 < \gamma < \beta$, and let x be any real number. With $z = x + i\gamma$, equation (5), in the form $F^+ - F^- = 0$, yields

(6) $$\int_R f(t)\, e^{-i(x+i\gamma)\,t}\, dt = 0.$$

Let $\psi(t) = f(t)\, e^{\gamma t}$. ψ is integrable on the real line, and equation (6) says that $\hat{\psi} \equiv 0$. Thus the uniqueness property of the Fourier transform [8, p 125] yields that $\psi = 0$ a. e., which says that $f = 0$ a. e. The proof is done.

In [2, p. 94], two power series with radius of convergence one are identified if their difference has radius of convergence larger than one. Thus the point of view that the Fourier-Carleman functions are analogues of power series, together with (1), suggests that two bounded functions f and g be identified if there is a positive number β for which

(7) $$\int_0^\infty |f(t) - g(t)|\, e^{\beta t}\, dt$$

is finite. This is an equivalence relation on $L^\infty(R)$; each class is uncountable, and the class containing f will be denoted $C(f)$. (Alternatively, let V be the vector space of bounded functions f for which (4) is finite for some positive β, and consider elements of $L^\infty(R)/V$.) If f and g are in the same class, then (1) says that $F^-(f-g, z)$ is holomorphic on a half-plane that is larger than $Im\ z < 0$.

Let f and g be in the same class, and suppose that f has the interval I as a spectral gap. Since $F^-(g, z) = F^-(f, z) + F^-(g-f, z)$, $F^-(g, z)$ also continues across I. However, it is not necessarily true that g has I as a spectral gap: it will be shown below in (8) that, if $\sigma(f)$ is discrete (for example, when f is periodic), then every other function in $C(f)$ has for spectrum the whole real line. In order for g to have I as a spectral gap, $F^-(g, z)$ must not only continue across I, but it must continue to something of a specific form. Thus the phenomenon of having a spectral gap must occur less frequently than that of analytic continuation. Theorems (8) and (10) comment on this frequency.

(8)   T h e o r e m. *Each $C(f)$ contains at most countably many functions whose spectrum is not the whole real line, and at most one whose spectrum is discrete. If $\sigma(f)$ is discrete, then $\sigma(g) = R$ for every other g in $C(f)$.*

P r o o f. If any one of the three conclusions is assumed to be false, then it follows that there are two distinct functions g and h in $C(f)$ having a common spectral gap I. The linearity of the Fourier-Carleman functions in

their first argument implies that I is spectral gap for g—h also. Since g and h are in $C(f)$, there is a positive number β for which

$$\int_0^\infty |g(t) - h(t)|\, e^{\beta t}\, dt$$

is finite. The uniqueness property (3) then says that g—$h = 0$ a. e., or that $g = h$ a. e., a contradiction that ends the proof.

Two examples will now be given to show that both extremes allowed by the phrase "...at most countably many..." can be realized. The first is an example of a function f for which $C(f)$ actually contains infinitely many members with (necessarily non-overlapping) spectral gaps; the second is an example in which every member of $C(f)$ has spectrum the whole real line.

E x a m p l e 1. Let $\hat{\psi}(t) = e^{-t^2}$ and let S be an even C^∞-function which takes the value one on $[-1/8, 1/8]$ and vanishes outside $[-1/4, 1/4]$. For each integer n, let $s_n(t) = s(t - n)$ and $\hat{\psi}_n(t) = 2^{-|n|}\hat{\psi}(t - n)$. The series

$$\sum_{n \in Z} s_n(t)\hat{\psi}_n(t)$$

is an even C^∞ — function each of whose derivatives is integrable. Thus [4, p. 13] the Fourier transform of this series goes to zero fast enough that it is itself integrable. Then the Inversion Theorem implies that the series is a Fourier transform, say $\hat{f}$. Since $\hat{f}$ is integrable and even, $f = \hat{\hat{f}}$, so that f is plainly bounded.

For each n in Z, let $g_n = f - \psi_n$. Since $\psi_n = 2^{-|n|}k\psi$, where $|k| = 1$, it is clear that each g_n is bounded and integrable. ψ and $\hat{\psi}$ are basically the same, so that

$$\int_0^\infty |f(t) - g_n(t)|\, e^{\beta t}\, dt = \int_0^\infty |\psi_n(t)|\, e^{\beta t}\, dt < \infty$$

for some positive number β. Thus $g_n \in C(f)$ for each integer n. $\hat{g}_n = \hat{f} - \hat{\psi}_n$ is zero only on $[n - 1/8, n + 1/8]$ and at some isolated points, so that $(n - 1/8, n + 1/8)$ is the only spectral gap that g_n has. This finishes the first example.

E x a m p l e 2. Let f be a Bohr almost periodic function with $\sigma(f) = R$. Bochner and Bohnenblust showed in [5] that, for each Fourier exponent λ of f,

(9) $|F^-(f, \lambda - iy)| \to \infty$ as $y \searrow 0$.

If g is in $C(f)$, then (1) says that $F^-(f - g, z)$ is holomorphic on a half-plane that is larger than $Im\, z < 0$. This fact, together with (9) and the triangle inequality, yields that

$$|F^-(g, \lambda - iy)| \to \infty \text{ as } y \searrow 0$$

for each Fourier exponent of f. But these are dense, so $\sigma(g) = R$ also.

Now, for each bounded function f and positive number β, let $N(f,\beta)$ be the collection of all bounded functions g for which

$$\int_0^\infty |f(t) - g(t)| \, e^{\beta t} \, dt$$

is finite. The topology that will be used on $L^\infty(R)$ is the one in which the sets $N(f, \beta)$ are the neighborhoods.

Although they will not be used in this paper, a few brief comments about this topology are in order. Redefine a neighborhood as follows: let $N^*(f, \beta)$ be the collection of all bounded functions g for which there exists a positive number ε such that

$$\int_0^\infty |f(t) - g(t)| \, e^{(\beta + \varepsilon)t} \, dt$$

is finite. Let

$$d(f, g) = \inf \left\{ \frac{1}{\gamma} \, \Big| \int_0^\infty |f(t) - g(t)| \, e^{\beta t} \, dt < \infty \right\};$$

then g is in $N^*(f, \beta)$ if and only if $d(f, g) < 1/\beta$. d is not a metric: $d(f, g)$ can be zero without f and g being equal a. e. Neither is it a pseudo-metric since it can take the value $+\infty$. But

$$D(f, g) = \frac{d(f, g)}{1 + d(f, g)}$$

is a pseudo-metric whose topology is equivalent to the one whose neighborhoods are the sets $N^*(f, \beta)$. Since $N^*(f, \beta) \subset N(f, \beta) \subset N^*(f, \beta/2)$, the topology on $L^\infty(R)$ that will be used is of equal strength with one generated by a pseudo-metric.

(10) T h e o r e m. *Relative to the topology on $L^\infty(R)$ determined by the neighborhoods $N(f, \beta)$, the set of functions having a spectral gap is a countable union of discrete sets.*

P r o o f. Let I be an open interval on the real line with rational end--points, and let F_I be the set (actually a subspace) of all bounded functions which have I as a spectral gap. The collection of bounded functions which have a spectral gap somewhere is the countable union $\cup_I F_I$. To see that each F_I is discrete, let f and g be distinct elements of F_I and suppose that g is in $N(f, \beta)$. $f{-}g$ has I as a spectral gap, so that, by the uniqueness property (3), $f = g$ a. e., a contradiction which ends the proof.

What is the mathematical use of such statements as Theorems (8) and (10)? If the spectrum of a certain bounded function is to be computed, if it is to be determined where a given power series continues across its circle of convergence, or if the transcendence of a specific constant is to be proved, such statements are of no use at all. They do, however, put in place a certain attitude, or expectation, despite a well-founded caution in their interpretation

[6, p. 33; note the cautious "...a strictement parler..."]. They can also make a qualitative statement about the nature of a problem. For example, since now, on the strength of (8) and (10), it can be said that most bounded functions have for their spectrum the whole real line, it is to be expected that the problem of reconstructing f from $\sigma(f)$—the problem of spectral synthesis [10, p. 184] —would be difficult to sort out, as indeed it has proven to be.

REFERENCES

[1] A.S. Besicovitch, *Almost Periodic Functions*, Dover Publications, Inc., 1954.

[2] L. Bieberbach, *Analytische Fortsetzung*, Ergebnisse der Matematik und Ihrer Grenzgebiete, Neue Folge, Heft 3, Springer-Verlag, 1955.

[3] R. Bieberich, *Bounded Functions with no Spectral Gaps*, Publ. Inst. Math. **26 (40)** (1979), 35—44.

[4] S. Bochner, *Lectures on Fourier Integrals*, Ann. Math. Stud. 42. Princeton University Press, 1959.

[5] S. Bochner, F. Bohnenblust, *Analytic Functions with Periodic Coefficients*, Ann. Math. 35, Second Series, (1934), 152—161.

[6] J. Hadamard, *La série de Taylor et son prolongement analytique*, Scientia, Series Physico--Mathématique, No. 12, Gauthier-Villars, 1901.

[7] E. Hille, *Analytic Function Theory, Volume I*, Introductions to Higher Mathematics, Blaisdell, 1962.

[8] Y. Katznelson, *An Introduction to Harmonic Analysis*, John Wiley and Sons, Inc. 1968.

[9] B. Ploeger, *The Spectrum of Certain Bounded Stepanoff Almost Periodic Functions*, Dissertation, The Ohio State University, 1975.

[10] W. Rudin, *Fourier Analysis on Groups*, Interscience Tracts in Pure and Applied Mathematics 12, Interscience (1967).

[11] H.S. Shapiro, *Topics in Approximation Theory*, Lecture Notes in Mathematics 187, Springer- Verlag (1971).

Department of Mathematics
Ball State University
Muncie, Indiana, 47306,
USA