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# A LOEB MEASURE APPROACH TO THE RIESZ REPRESENTATION THEOREM

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**Abstract.** A short non-standard analysis proof of the Riesz representation theorem is given.

The concept of the Loeb-measure proved to be very useful in constructions of various kinds of limit measures. The papers of Anderson [1], Loeb [5] and Lindstrøm [3], to mention just a few, are typical examples where this technique was successfully applied.

The cornerstone of the metod is the following theorem.

T h e o r e m 1. (Loeb [4]) *Let* $(X, \mathcal{A}, \mu)$ *be an internal finitely additive measure space and* ${}^\circ\mu(X) < +\infty$, *where* ${}^\circ\mu(A) := \mathrm{st}(\mu(A))$. *Then* ${}^\circ\mu$ *has a unique extension* $L(\mu)$ *to the completion* $L(\mathcal{A})$ *of the σ-algebra generated by* $\mathcal{A}$. *More precisely, for all* $A \in L(\mathcal{A})$, $L(\mu)(A) = \sup\{{}^\circ\mu(B) \mid B \in \mathcal{A}, B \subset A\} = \inf\{{}^\circ\mu(C) \mid C \in \mathcal{A}, C \supset A\}$.

We shall also need the folowing result of T. Lindstrøm, more precisely, a simple special case when $X$ is compact. Recall that Radon measure $\nu$ on a topological space $X$ is a complete measure defined on a σ-algebra extending the Borel algebra on $X$, such that for all sets $B \in \mathcal{B}$

$$\nu(B) = \sup\{\nu(K) \mid K \subset B,\ K \text{ compact}\} = \inf\{\nu(O) \mid B \subset O,\ O \text{ open}\}.$$

T h e o r e m 2. (T. Lindstrøm [2]) *Let* $({}^*X, \mathcal{A}, \mu)$ *be an internal finite additive measure space, where* ${}^\circ\mu({}^*X) < +\infty$. *Assume that there is a subbasis* $\tau$ *for the topology on* $X$ *closed under finite unions, such that for all* $O \in \tau$, ${}^*O \in L(\mathcal{A})$. *Then* $\mathrm{st}^{-1}(K) \in L(\mathcal{A})$ *for every compact* $K$ *and* $L(\mu)(\mathrm{st}^{-1}(K)) = \inf\{L(\mu)({}^*O) \mid K \subset O, O \in \tau\}$. *If for each* $\varepsilon > 0$ *there is a compact* $K \subset X$ *such that* $L(\mu)(\mathrm{st}^{-1}K) > L(\mu)({}^*X) - \varepsilon$, *then* $\nu$ *defined by* $\nu(A) = L(\mu)\,\mathrm{st}^{-1}A$ *is a Radon measure.*

T h e o r e m 3. The Riesz Representation Theorem: *Let* $X$ *be a compact Hausdorff space and* $L: C(X) \to R$ *a positive linear function. Then there exists a Radon measure* $\nu$ *on* $X$ *such that* $L(f) = \int_X f\, d\nu$.

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P r o o f. Let $(\mathcal{P}, \leqslant)$ be the following partial order associated with space $X$.

$$a\in\mathcal{P} \text{ is equivalent with } a=(\mathcal{U}, \mathcal{F})$$

where $\mathcal{U}$ it a finite open cover of $X$ and $\mathcal{F}$ finite partition of unity subordinated to $\mathcal{U}$. Further on we assume that $\mathcal{U}$ and $\mathcal{F}$ are in 1-1 correspondence $\varphi: \mathcal{U}\to\mathcal{F}$ such that $\operatorname{supp}\varphi(V)\subset V$ for every $V\in\mathcal{U}$. The order $\leqslant$ is defined as follows. If $a=(\mathcal{U}, \mathcal{F})$, $b=(\mathcal{V}, \mathcal{G})$ then

(1) $$a\leqslant b \leftrightarrow \text{(i) and (ii) where}$$
$$\text{(i) } \forall U\in\mathcal{U}\, \exists S\subset\mathcal{V}\ (U=\text{union } S)$$
$$\text{(ii) } \forall O\subset X \text{ open set } \Sigma\{f\in\mathcal{F} \mid \operatorname{supp}(f)\subset O\}\leqslant$$
$$\leqslant\Sigma\{g\in\mathcal{G} \mid \operatorname{supp}(g)\subset O\},\ \operatorname{supp}(f):=\operatorname{cl}\{x \mid f(x)>0\}.$$

Let $\pi:\mathcal{U}\to X$ be a choice function associated with $a=(\mathcal{U}, \mathcal{F})\in\mathcal{P}$ which has the property $\forall V\in\mathcal{U}\quad \pi V\in\operatorname{supp}\varphi(V)$. We need the following

C l a i m. $(\mathcal{P}, \leqslant)$ is a directed partially ordered set.

Indeed, if $a=(\mathcal{U}, \mathcal{F})$, $b=(\mathcal{V}, \mathcal{G})$ are in $\mathcal{P}$ then $c=(\mathcal{W}, \mathcal{H})$ defined by $\mathcal{W}=\{U\cap V \mid U\in\mathcal{U}, U\in\mathcal{V}\}$, $\mathcal{H}=\{f\cdot g \mid f\in\mathcal{F}, g\in\mathcal{G}\}$ is a common refinement of $a$ and $b$.

By saturation there exists $\bar{a}=(\overline{\mathcal{U}}, \overline{\mathcal{F}})\in{}^*\mathcal{P}/\mathcal{P}$ which refines all elements of $\mathcal{P}$. $\overline{\mathcal{U}}$ is a hyperfinite set and let $\mathcal{A}$ be the hyperfinite algebra generated by $\overline{\mathcal{U}}$. Let us note that $^*O\in\mathcal{A}$ for every open $O\subset X$, as a consequence of (ii) in (1). Indeed, $O\in\mathcal{U}$ for same $a=(\mathcal{U}, \mathcal{F})$.

Let $\bar{\pi}:\overline{\mathcal{U}}\to{}^*X$ be the choice function associated with $\bar{a}=(\overline{\mathcal{U}}, \overline{\mathcal{F}})$. For $A\in\mathcal{A}$ let

$$\mu(A):={}^*\Sigma\{{}^*L(f) \mid \exists V\in\overline{\mathcal{U}}\ (\bar{\pi}V\in A\ \&\ ({}^*\operatorname{supp})(f)\subset V)\}.$$

$({}^*X, \mathcal{A}, \mu)$ is an internal measure space, hence by Theorems 1. and 2. $\nu=L(\mu)\operatorname{st}^{-1}$ is a Radon measure. Let us prove that $\nu$ is the desired measure.

Let $O\subset X$ be open and $f\in C(X)$, such that $0\leqslant f\leqslant 1$ and $\operatorname{supp}(f)\subset O$. Then

(2) $$L(f)\leqslant\nu(O).$$

To prove this, let us note that $(\{O, X\}, \{f, 1-f\})\in\mathcal{P}$ and, if $O\neq X$, $\operatorname{supp}(1-f)\not\subset O$ which by (ii) in (1) implies ${}^*f\leqslant{}^*\Sigma\{h\in\overline{\mathcal{F}} \mid ({}^*\operatorname{supp})(h)\subset{}^*O\}$ $L(f)={}^*L({}^*f)\leqslant{}^*\Sigma\{{}^*L(h) \mid h\in\overline{\mathcal{F}}, {}^*\operatorname{supp}(h)\subset{}^*O\}\leqslant\mu({}^*O)$ by the definition of $\mu$.

The same inequality holds if $O$ is replaced by an open set $G$ such that $\operatorname{supp}(f)\subset G\subset\operatorname{cl} G\subset O$ Hence,

$$L(f)\leqslant\operatorname{st}\mu({}^*G)\leqslant\operatorname{st}\mu({}^*\operatorname{cl} G)\leqslant L(\mu)\operatorname{st}^{-1}\operatorname{cl} G\leqslant L(\mu)\operatorname{st}^{-1} O=\nu(O).$$

Now, let $f\in C(X)$ be any function. Clearly, $L(1_X)=\nu(X)$. Thus, we can assume, without loss of generality, that $f(X)\subset[s, t]$ for $s\geqslant 0$. Let $s=x_0<x_1<\cdots<x_n=t$ be a subdivision of $[s, t]$ such that $|x_{i+1}-x_i|<\varepsilon$, $0\leqslant i\leqslant n-1$, and $\nu(f^{-1}(x_i))=0$ which is possible by the $\sigma$-additivity of the measure $\nu$. Choose $O_i\supset F_i$ such that $\nu(O_i\setminus F_i)<\varepsilon/2^i x_i$ where $F_i=f^{-1}[x_{i-1}, x_i]$ $1\leqslant i\leqslant n$. Let $0\leqslant f_i\leqslant 1$ be chosen so that $f_i(F_i)=\{1\}$ and $\operatorname{supp}(f_i)\subset O_i$.

$$\text{Clearly, } f \leqslant \sum_{i=1}^{n} x_i f_i. \text{ Thus } L(f) \leqslant \sum_{i=1}^{n} x_i L(f_i) \leqslant$$

$$\leqslant \sum_{i=1}^{n} x_i \nu(O_i) \leqslant \sum_{i=1}^{n} x_i \nu(F_i) + \varepsilon.$$

Knowing that $F_i \cap F_j$ has $\nu$-measure zero for $i \neq j$, one recognizes on the right side, a sum which is arbitrary good approximation of $\int_X f\,d\nu$. Hence $L(f) \leqslant \int_X f\,d\nu$ for every $f \in C(X)$. Applying the last inequality to the function $-f$ we get the desired equality $L(f) = \int_X f\,d\nu$.

*Note added in proof.* The author has been informed that profesor Peter Loeb obtained a short non-standard analysis proof of the Riesz representation theorem based on different ideas.

## REFERENCES

[1] R.M. Anderson, *A Non-standard Representation of Brownian Motion and Ito Integration*, Isreal J. Math. **25** (1976) 15—46

[2] T.L. Lindstrøm, *Applications of Nonstandard Analysis to Probability Theory*, notes from the lectures given at the University of Wisconsin, Madison, 1981.

[3] T.L. Lindstrøm, *A Loeb-measure Approach to Theorems by Prohorov, Sazonov and Gross*, preprint.

[4] P.A. Loeb, *Conversion from Non-standard to Standard Measure Spaces and Applications in Probability Theory*, Trans. Amer. Math. Soc., **211** (1975), 113—122.

[5] P.A. Loeb, *Weak Limits of Measures and the Standard Part Map*, Proc. Amer. Math. Soc., **77** (1979), 128—135.

[6] K.D. Stroyan, W.A.J. Luxemburg, *Introduction to the Theory of Infinitesimals*, Academic Press, 1976.

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