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ON THE PROXIMATE TYPE OF AN ENTIRE FUNCTION

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Abstract. The proximate type T and lower proximate type t of an entire function $f(z)= \sum_{n=0}^{\infty} a_n z^n$ with respect to the proximate order $\rho(r)$ are defined as:

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \log M(r)/r^{\rho(r)} = \begin{matrix}T\\ t\end{matrix} \qquad 0\leqslant t\leqslant T\leqslant\infty,\ M(r)=\max_{|z|=r} |f(z)|.$$

We have obtained some of the growth properties of the entire function $f(z)$ with the help of the proximate order $\rho(r)$ and proximate type T. Our main aim in this paper is to prove the following theorem: the proximate type of the derivative of $f(z)$ is the same as that of $f(z)$ with respect to the proximate order $\rho(r)$.

1. Introduction. Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ be an entire function of order ρ, lower order λ and type P. Denoting with $M(r)$ the $\max_{|z|=r} f|(z)|$, the type of $f(z)$ is defined by

$$\lim_{r\to\infty} \sup \log M(r)/r^{\rho} = P, \quad 0\leqslant P\leqslant\infty. \tag{1.1}$$

Further, let

$$\mu(r)=\sup\{|a_n| r^n : n\geqslant 0\}, \quad \nu(r)=\max\{n : |a_n| r^n=\mu(r)\}.$$

Then $\mu(r)=|a_{\nu(r)}| r^{\nu(r)}$. Similarly, if $\mu_s(r)$ and $\nu_s(r)$, $(s=1, 2, 3, \ldots)$ denote the maximum term and its rank for the function $f^{(s)}(z)$ the sth derivative of $f(z)$, respectively, then $\mu(r)$, $\mu_s(r)$, $\nu(r)$ and $\nu_s(r)$ are all positive and non-decreasing functions of r. We write $\mu_0(r)\equiv\mu(r)$ and $\nu_0(r)\equiv\nu(r)$.

It is to be noted that $\log\mu(r)$ is an increasing convex function of $\log r$. In fact [6, p. 31]

$$\log\mu(r)=\log\mu(r_0)+\int_{r_0}^{r} \frac{\nu(x)}{x}\, dx, \quad r>r_0. \tag{1.2}$$

A function $\rho(r)$ satisfies the conditions

(1.3) $$\lim_{r\to\infty} \rho(r)=\rho \quad \text{and}$$

(1.4) $$\lim_{r\to\infty} r\rho'(r)\log r=0$$

is called a proximate order [1, p. 54].

The proximate type T and lower proximate type t of $f(z)$ with respect to the proximate order $\rho(r)$ are defined as:

(1.5) $$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \log M(r)/r^{\rho} = \begin{matrix}T\\ t\end{matrix}, \quad 0\leqslant t\leqslant T\leqslant\infty.$$

An entire function $f(z)$ is said to be of irregular grotwh, if $\rho\neq\lambda$ and it has been shown by Srivastava [4, p. 345] that lower type of an entire function of irregullar growth of finite order is zero, that is

(1.6) $$\lim_{r\to\infty}\inf \log M(r)/r^{\rho}=0, \quad 0\leqslant\lambda<\rho<\infty.$$

In such a case if the quantity

(1.7) $$\lim_{r\to\infty}\inf \log M(r)/r^{\lambda(r)}=t_{\lambda}$$

is different from zero and infinity, then t_{λ} is defined to be the λ-type of $f(z)$ with respect to the comparison function $\lambda(r)$, which possesses the following properties:

(1.8) $$\lim_{r\to\infty} \lambda(r)=\lambda \quad \text{and}$$

(1.9) $$\lim_{r\to\infty} r\lambda'(r)\log r=0.$$

Our aim in this paper is to study a few growth properties of the entire function $f(z)$ with the help of proximate order $\rho(r)$ and proximate type T. Throughout this paper we shall assume $0<\rho<\infty$. We prove the following theorems:

2. Theorems. Theorem 1. *We find*

(2.1) $$\lim_{r\to\infty} \{L(r+b)/L(r)\}^{r/b}=1, \quad b>0,$$

where $L(r)=r^{\rho(r)-\rho}$.

Theorem 2. *Let* $F(r)$ *be any function bounded on each finite interval and increasing monotonically to infinity for* $r>r_0$, *and*

(2.2) $$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} F(r)/r^{\rho(r)} = \begin{matrix}\alpha\\ \beta\end{matrix}, \quad 0\leqslant\beta\leqslant\alpha\leqslant\infty,$$

then

$$\limsup_{r\to\infty}\left\{\frac{1}{r^{\rho(r)}}\int_{r_0}^{r}\frac{F(x)}{x}\,dx\right\}\leqslant\frac{\alpha}{\rho}, \tag{2.3}$$

and

$$\liminf_{r\to\infty}\left\{\frac{1}{r^{\rho(r)}}\int_{r_0}^{r}\frac{F(x)}{x}\,dx\right\}\geqslant\frac{\beta}{\rho}. \tag{2.4}$$

Theorem 3. *Let $f(z)$ be an entire function of order ρ. Then with respect to the proximate order $\rho(r)$, $f(z)$ is of* (i) *minimal proximate type when $H=0$,* (ii) *maximal proximate type when $h=\infty$,* (iii) *normal proximate type when $H\neq\infty$ and $h\neq 0$, where*

$$\lim_{r\to\infty}\frac{\sup}{\inf}\,\nu(r)/r^{\rho(r)}=\frac{H}{h},\quad 0\leqslant h\leqslant H\leqslant\infty, \tag{2.5}$$

Theorem 4. *Let T and t be the proximate type and lower proximate type, respectively, of $f(z)$ with respect to the proximate function $\rho(r)$. If ρ be the order of $f(z)$ and*

$$\lim_{r\to\infty}\frac{\sup}{\inf}\,M'(r)/M(r)\,r^{\rho(r)}=\frac{\gamma}{\delta},\quad 0\leqslant\delta\leqslant\gamma\leqslant\infty, \tag{2.6}$$

where $M'(r)$ is the derivative of $M(r)$ which exists for almost all values of r, then

$$\delta\leqslant\rho\, t\leqslant\rho\, T\leqslant\gamma. \tag{2.7}$$

Theorem 5. *The proximate type of the derivative of $f(z)$ is the same as that of $f(z)$ with respect to the proximate order $\rho(r)$.*

Theorem 6. *If ρ be the order and T be the proximate type with respect to the proximate order $\rho(r)$ of an entire function $f(z)$ and $\lim_{r\to\infty}\nu(r)/r^{\rho(r)}$ exists, then*

$$\lim_{r\to\infty}\{r\,[\mu_m(r)/\mu(r)\}^{1/m}\,r^{-\rho(r)}]=\rho\, T. \tag{2.8}$$

Theorem 7. *The λ-type of the derivative of $f(z)$ is same as that of $f(z)$ with respect to $\lambda(r)$, provided $|a_n/a_{n+1}|$ forms a non-decreasing function of n for $n>n_0$.*

3. Proofs of theorems. Proof of Theorem 1. Let us set $\varphi(r)=\rho(r)\log r$, then

$$\begin{aligned}\log\{L(r+b)/L(r)\}&=(\rho(r+b)-\rho)\log(r+b)-(\rho(r)-\rho)\log r\\&=(\varphi(r+b)-\varphi(r))-\rho\log(1+b/r)\end{aligned}$$

$$= b\,\varphi'(k) - b\,\rho(1 - o(1))/r$$

$$= b\{\rho(r)/r + \rho'(r)\log r\} - b\,\rho\,(1 - o(1))/r,$$

$$r_0 < r < k < r + b$$

or, $r\,(\log\{L(r+b)/L(r)\}/b = \rho(r) + r\,\rho'(r)\log r - \rho\,(1 - o(1)).$

From (1.3) and (1.4), we have

$$\rho(x) - \rho < \varepsilon/2 \text{ and } |x\,\rho'(x)\log x| < \varepsilon/2,$$

where ε is an arbitrarily taken small positive number. Using this, we get

$$\lim_{r\to\infty} rb^{-1}\log\{L(r+b)/L(r)\} = 0.$$

Hence, $\lim\limits_{r\to\infty} \{L(r+b)/L(r)\}^{r/b} = 1.$

This completes the proof of Theorem 1.

Corollary 1. *For* $r > r_0$, *we have*

$$1 - \varepsilon < \{(r+b)^{\rho(r+b)-\rho}\,r^{\rho-\rho(r)}\}^{r/b} < 1 + \varepsilon,\ \ \varepsilon > 0.$$

To prove Theorem 2 we need the following lemma.

Lemma 1. $\displaystyle\int_{r_0}^{r} x^{\rho(x)-1}\,dx = \frac{1}{r}\,r^{\rho(r)} + o\,(r^{\rho(r)}),\ \ r > r_0.$

Proof. We have

$$\int_{r_0}^{r} x^{\rho(x)-1}\,dx = \int_{r_0}^{r} x^{\rho(x)-\rho}\,x^{\rho-1}\,dx$$

$$= \left[x^{\rho(x)-\rho}\,\frac{x^{\rho}}{\rho}\right]_{r_0}^{r} - \frac{1}{\rho}\int_{r_0}^{r} x^{\rho(x)-1}\{\rho(x) - \rho + x\,\rho'(x)\log x\}\,dx$$

$$= \frac{1}{\rho}\,r^{\rho(r)} + O(1) - o(1)\int_{r_0}^{r} x^{\rho(x)-1}\,dx$$

or, $\displaystyle(1 + o(1))\int_{r_0}^{r} x^{\rho(x)-1}\,dx = \frac{1}{\rho}\,r^{\rho(r)} + O(1).$

Thus, finally, we obtain

$$\int_{r_0}^{r} x^{\rho(x)-1}\, dx = \frac{1}{\rho} r^{\rho(r)} + o\,(r^{\rho(r)}).$$

Proof of Theorem 2. From (2.2), we have, for any $\varepsilon > 0$ and $r > r_0$, $F(r) < (\alpha+\varepsilon)\, r^{\rho(r)}$. Therefore,

$$\int_{r_0}^{r} \frac{f(x)}{x}\, dx = \int_{r_0}^{r'} \frac{F(x)}{x}\, dx + \int_{r'}^{r} \frac{F(x)}{x}\, dx, \ \leqslant O(1) + (\alpha+\varepsilon) \int_{r'}^{r} x^{\rho(x)-1}\, dx$$

$$= O(1) + (\alpha+\varepsilon) \left\{ \frac{1}{\rho} r^{\rho(r)} + o\,(r^{\rho(r)}) \right\}, \quad r_0 < r' < r$$

or,

$$\limsup_{r\to\infty} \left\{ \frac{1}{r^{\rho(r)}} \int_{r_0}^{r} \frac{F(x)}{x}\, dx \right\} \leqslant \frac{\alpha}{\rho}.$$

Similarly, considering the inequality

$$F(r) > (\beta - \varepsilon)\, r^{\rho(r)}, \quad r > r_0, \quad \varepsilon > 0,$$

we can prove (2.4).

Corollary 2. *If* $\lim\limits_{r\to\infty} F(r)/r^{\rho(r)} = B$, *then*

$$\lim_{r\to\infty} \left\{ \frac{1}{r^{\rho(r)}} \int_{r_0}^{r} \frac{F(x)}{x}\, dx \right\} = \frac{B}{\rho}, \quad r > r_0.$$

Proof of Theorem 3. It is well known that the function $\nu(r)$ is bounded in finite intervals, has an enumerable set of discontinuities and changes values at these discontinuities only. Also, for functions of finite order $\log M(r) \sim \log \mu(r)$. Hence, from (1.2) and (2.3), we get

$$T = \limsup_{r\to\infty} \log M(r)/r^{\rho(r)} = \limsup_{r\to\infty} \log \mu\,(r)/r^{\rho(r)}$$

$$= \limsup_{r\to\infty} \left[\frac{1}{r^{\rho(r)}} \left\{ \log \mu\,(r_0) + \int_{r_0}^{r} \frac{\nu(x)}{x}\, dx \right\} \right]$$

$$= \limsup_{r\to\infty} \left\{ \frac{1}{r^{\rho(r)}} \int_{r_0}^{r} \frac{\nu\,(x)}{x}\, dx \right\} \leqslant \frac{H}{\rho},$$

or,

(3.1) $$\rho T \leqslant H.$$

Similarly, proceeding for lower proximate type t and applying (2.4), we obtain

(3.2) $$\rho t \geqslant h.$$

Combining (3.1) and (3.2), we find

(3.3) $$h \leqslant \rho t \leqslant \rho T \leqslant H.$$

All the three results (i), (i) and (iii) of Theorem 3 now follow from (3.3).

Corollary 3. *If* $\lim_{r\to\infty} \nu(r)/r^{\rho(r)}$ *exists and is equal to* D, *then* $H=h=D$. *It follows from* (3.3) *that* $\rho t = \rho T = D$, *that is*

(3.4) $$\rho T = \lim_{r\to\infty} \nu(r)/r^{\rho(r)}$$

and in this case

(3.5) $$t = T = \lim_{r\to\infty} \log M(r)/r^{\rho(r)}.$$

Thus existence of $\lim_{r\to\infty} \nu(r)/r^{\rho(r)}$ implies the existence of $\lim_{r\to\infty} \log M(r)/r^{\rho(r)}$.

Proof of Theorem 4. It is known [6, p. 27] that

(3.6) $$\log M(r) = \log M(r_0) + \int_{r_0}^{r} \frac{W(x)}{x}\, dx, \quad r > r_0$$

where $W(r)$ is a positive, indefinitely increasing function. Hence differentiating we get $M'(r)/M(r) = W(r)/r$. But $M'(r)/M(r)$ is bounded on each finite interval. Hence (2.6) follows from (2.3) and (2.4).

Proof of Theorem 5. We have [3, p. 42]

(3.7) $$\limsup_{n\to\infty} \varphi(n)\, |a_n|^{1/n} = (e\rho T)^{1/\rho},$$

where $\varphi(t)$ is the unique (for $t > t_0$) solution of the equation $t = r^{\rho(r)}$.

Let T_m be the proximate type of the mth derivative $f^{(m)}(z)=\sum_{n=0}^{\infty}(n+m)$ $(n+m-1)\cdots(n+1)a_{n+m}z^n$ with respect to proximate order $\rho(r)$. It is well known that the order of $f^{(m)}(z)$ is equal to that of $f(z)$. Hence

$$(e\rho T_m)^{1/\rho}=\limsup_{n\to\infty}\varphi(n)\{(n+m)(n+m-1)\cdots(n+1)\,|\,a_{n+m}|\}^{1/n}$$

$$=\limsup_{n\to\infty}\varphi(n)\,|\,a_n|^{1/n}\{(n+m)(n+m-1)\cdots(n+1)\}^{1/n}\,|\,a_{n+m}/a_n|^{1/n}$$

$$=\limsup_{n\to\infty}\varphi(n)\,|\,a_n|^{1/n}=(e\rho T)^{1/\rho}.$$

This gives,

(3.8) $$T_m=T.$$

Before we start the actual proof of the next theorem, we consider the following lemma which will be needed in this proof.

Lemma 2. *For* $m=1, 2, 3, \ldots$

$$\lim_{r\to\infty}\nu(r)/r^{(\rho)}=\lim_{r\to\infty}\nu_m(r)/r^{\rho(r)}.$$

Proof. We have

$$\log\mu(r)=\log\mu(r_0)+\int_{r_0}^{r}\frac{\nu(x)}{x}\,dx$$

and $\log\mu_m(r)=\log\mu_m(r_0)+\int_{r_0}^{r}\frac{\nu_m(x)}{x}\,dx, \quad r>r_0.$

Also, $\lim_{r\to\infty}\nu_m(r)/r^{\rho(r)}$ exists for all non-negative integers m. Hence from (3.4), we obtain

(3.9) $$\rho T=\lim_{r\to\infty}\nu(r)/r^{\rho(r)} \quad \text{and}$$

(3.10) $$\rho T_m=\lim_{r\to\infty}\nu_m(r)/r^{\rho(r)}.$$

Lemma 2 now follows from (3.9) and (3.10).

Proof of Theorem 6. The proof of this theorem can be obtained by using Lemma 2 and the following result [5. p. 276]:

$$\nu(r)/r\leqslant\{\mu_m(r)/\mu(r)\}^{1/m}\leqslant\nu_m(r)/r.$$

Proof of Theorem 7. If $F(x)$ is the unique solution of the equation $x = r^{\lambda(r)}$, for $x > x_0$, then the λ-type t_λ of the entire function $f(z)$ is given by [2, p. 76]

$$\liminf_{n\to\infty} \{F(n) \, |a_n|^{1/n}\} = (t_\lambda \lambda e)^{1/\lambda}, \tag{3.11}$$

provided $|a_n/a_{n+1}|$ is a non-decreasing function of n for $n > n_0$.

Now, proceeding as for Theorem 4, we can prove the required result by using (3.11) instead of (3.7). The details are omitted.

REFERENCES

[1] M.L. Cartwright, *Integral functions*, Cambridge Tract, C.U.P. 1956.

[2] Kuldip Kumar, *A note on entire functions of irregular growth*, Ganita, **28**(1977), 75—78.

[3] B.Ja. Levin, *Distribution of zeros of entire functions*, Trans. AMS **5**(1964),

[4] R.S.L. Srivastava, Prem Singh, *On the λ-type of an entire function of irregular growth*, Arch. Math. **27**(1966), 342—346.

[5] S.K. Vaish, G.S. Srivastava, *On the maximum term any its rank of an entire function of slow growth. I*, Rev. Roum. Math. Pur. Appl. **25**(1980), 277—282.

[6] G. Valiron, *Lectures on the general theory of integral functions*, Chelsea Publ. Co., New York, 1949.

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