PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 32 (46) 1982, pp. 165—166

THREE PROPOSITIONS EQUIVALENT TO THE AXIOM OF CHOICE

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Abstract. Three propositions concerning mappings of a set into itself, two of which were recently proved by D. Banković, are shown to be equivalent to the axiom of choice.

Let $f: S \xrightarrow{\text{onto}} R$, where $R \subseteq S$. An easy consequence of the axiom of choice is

Proposition 0. *Every mapping g of S into R has the form $f \circ h$ for some $h: S \to S$.*

Proof. Choose $h(x) \in f^{-1}(\{g(x)\})$ for each $x \in S$.

Two refinements of this proposition were proved by D. Banković [1]:

Proposition 1. *Let f be as above. Then the mappings of S onto R are precisely the mappings $f \circ h$ with $h: S \to S$ satisfying $(\forall y \in R)(\exists x \in S)\, h(x) \in \in f^{-1}(\{y\})$.*

Proposition 2. *Let f be as above. Then the retractions of S onto R (i.e., the mappings $g: S \to R$ satisfying $g(x) = x$ for all $x \in R$) are precisely the mappings $f \circ h$ with $h: S \to S$ satisfying $(\forall y \in R)\, h(y) \in f^{-1}(\{y\})$.*

We now show, conversely, that each of these three propositions implies the axiom of choice. Let $M = \{A, B, C, \cdots\}$ be a family of nonempty, pairwise disjoint sets and let $L = \bigcup M = A \cup B \cup C \cup \cdots$ Assume that L and M are disjoint. By the axiom of regularity, $M \notin L \cup M$, i.e., $\{M\}$ is disjoint from L and M. Let $S = L \cup M \cup \{M\}$, $R = M \cup \{M\}$, and define $f: S \xrightarrow{\text{onto}} R$ by

$f(p) =$ the set X in M such that $p \in X$, if $p \in L$;

$f(X) = M$ if $X \in M$;

$f(M) = M.$

Define $g: S \xrightarrow{\text{onto}} R$ by

$$g(p) = M \text{ if } p \in L;$$

$$g(X) = X \text{ if } X \in M;$$

$$g(M) = M.$$

Observe that g is a retraction of S onto R. Thus each of Propositions 0, 1, and 2 implies that $g = f \circ h$ for some $h: S \to S$. For each $X \in M$, we have $X = g(X) = f(h(X))$, and inspection of the definition of f shows that $h(X)$ must be an element of X. Therefore $\{h(X): X \in M\}$ is a choice set for M.

In case L and M are not disjoint, we obtain a choice set for M by carrying out a similar argument with S replaced by $S' = L \cup M' \cup \{M\}$, where M' consists of the sets $\langle M, X \rangle$ for $X \in M$. The sets L, M' and $\{M\}$ are disjoint and M' is in one-to-one correspondence with M.

REFERENCES

[1] D. Banković, *On general and reproductive solutions of arbitrary equations*, Publ, Inst Math. (Beograd) **26 (40)** (1979), 31—33.

Mathematical Reviews
Ann Arbor, Michigan 48109
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(Received 01 02 1981)