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CONFORMAL INFINITESIMAL TRANSFORMATIONS OF ALMOST PARACONTACT METRIC STRUCTURES

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Abstract. We study some properties of conformal infinitesimal transformations of almost paracontact metric structure.

1. Introduction. Let us consider an $(2n+1)$ dimensional real differentiable manifold M^{2n+1} with a fundamental tensor field F of type (1.1), a fundamental vector field T and a 1-form A, such that for every vector field X, we have [1]

(1.1a) $$F^2 = I - A \otimes T,$$

(1.1b) $$FT = 0,$$

(1.1c) $$A(\overline{X}) = 0, \quad \overline{X} \overset{\text{def}}{=} F(X),$$

(1.1d) $$A(T) = 1 \quad \text{and} \quad \text{rank}(F) = 2n,$$

where I is the identity endomorphism of the tangent bundle of M^{2n+1}. Then M^{2n+1} is called an almost paracontact manifold. An almost paracontact manifold M^{2n+1} is said to be an almost paracontact metric manifold if a Riemannian metric G satisfies [1]

(1.2a) $$G(FX, FY) = G(X, Y) - A(X)A(Y),$$

(1.2b) $$G(T, X) = A(X).$$

The fundamental 2-form $'F$ of the structure, defined as $'F(X, Y) = G(FX, Y)$ satisfies $'F(X, Y) = {'F}(Y, X) = {'F}(\overline{X}, \overline{Y})$.

An almost paracontact metric structure is called a paracontact metric structure if

$$2\,'F(X, Y) = (D_X A)(Y) + (D_Y A)(X) = (L_T G)(X, Y),$$

where D is the Riemannian connection induced by G on M^{2n+1} and L_T denotes the Lie derivative along T.

A paracontact Riemannian manifold whose 1-form A is closed, that is $(D_X A)(Y) - (D_Y A)(X) = 0$, and

$$(D_X F)(Y) = 2A(X)A(Y)T - G(X, Y)T - A(Y)X,$$

is called a normal paracontact Riemannian manifold [2].

2. A differentiable vector field X on M^{2n+1} is called an infinitesimal transformation if there exists a nonconstant differentiable function ρ on M^{2n+1} such that [4] $L_X G = \rho G$. In particular if $\rho = 0$ on M^{2n+1}, X is called an infinitesimal isometry and if $\rho = \text{constant} \neq 0$ on M^{2n+1}, it is called an infinitesimal homothety.

A diffeomorphism $F: M^{2n+1} \to M^{2m+1}$ between two almost paracontact metric manifolds will be called a conformal transformation if it induces a conformal change of the two structures and an infinitesimal transformation X on M^{2n+1} will be called conformal if the corresponding local one parameter group consists of conformal transformations.

A conformal change of the almost paracontact metric structures (F, T, A, G) and $(\tilde{F}, \tilde{T}, \tilde{A}, \tilde{G})$ on M^{2n+1} is a change of the form [5]

$$\tilde{F} = F, \quad \tilde{A} = e^{\sigma} A, \quad \tilde{T} = e^{-\sigma} T, \quad \tilde{G} = e^{2\sigma} G, \tag{2.2}$$

where σ is a differentiable function on M^{2n+1}. This change preserves the relation (1.1) and (1.2).

It is rather classical that, for a conformal infinitesimal transformation U, we shall have

$$L_U F = 0, \quad L_U T = \lambda T, \quad L_U A = \mu A, \quad L_U G = \rho G, \tag{2.3}$$

where λ, μ, ρ are nonconstant differentiable functions. If we apply the operator L_U to the equation (1.1d) and (1.2a) and if we use in our computations (2.3), it is easy to obtain $\mu = -\lambda$ and $\rho = 2\mu$. Thus it follows that the vector field U is a conformal infinitesimal transformation if and only if it satisfies

$$L_U F = 0, \quad L_U T = \lambda T, \quad L_U A = -\lambda A, \quad L_U G = -2\lambda G. \tag{2.5}$$

Proposition 2.1. *A vector field U is a conformal infinitesimal transformation if and only if $L_U F = 0$, $L_U G = -2\lambda G$.*

Proof. Taking the Lie derivative of $F(X)$, we get

$$(L_U F)(X) = L_U(F(X)) - F(L_U X), \tag{2.7}$$

and replacing X by T, we get $F(L_U T) = 0$. Since $\text{rank}(F) = 2n$, $L_U T = \mu T$. Similarly, from $A(F) = 0$, we get $(L_U A) \circ F = 0$, which implies $L_U A = \nu A$, where ν is a nonconstant differentiable function.

Finally taking the Lie derivative of $A(T) = 1$ and $G(T, X) = A(X)$ along U, we get $\mu = \lambda = -\nu$.

Proposition 2.2. *A vector field U is a conformal infinitesimal transformation if and only if for any two functions* μ *and* λ,

$$L_U G = -2\lambda G \qquad L_U{}'F = 2\mu{}'F \tag{2.8}$$

then $\mu = -\lambda$.

Proof. Taking the Lie derivative of $'F(X, Y)$, we get

$$(L_U{}'F)(X, Y) = (L_U G)(FX, Y) + G((L_U F)(X), Y) \tag{2.9}$$

which due to (2.5) gives

$$(L_U{}'F)(X, Y) = -2\lambda G(FX, Y) = -2\lambda{}'F(X, Y)$$

i.e. $(L_U{}'F) = -2\lambda{}'F = 2\mu{}'F$, because $\mu = -\lambda$.

Conversely from (2.8) and (2.9), we get

$$L_U F = 2(\lambda + \mu) F. \tag{2.10}$$

Now it is easy to establish the following auxiliary formula,

$$L_U(F^3) = (L_U F)\circ(F^2) + F\circ(L_U F)\circ F + F^2\circ(L_U F) \tag{2.11}$$

Then using the para F-structure, $F^3 - F = 0$, and (2.10) in formula (2.11), we get $\lambda + \mu = 0$ i.e. $\lambda = -\mu$. Thus from (2.10) we get $L_U F = 0$.

Proposition 2.3. *In paracontact metric manifold* $M^{2n+1}(n \geq 1)$, *every conformal infinitesimal transformation does not admit, in general, an infinitesimal automorphism.*

Proof. In paracontact metric manifold, we have

$$2{}'F(X, Y) = (L_T G)(X, Y). \tag{2.13}$$

Taking the Lie derivative of the above equation, we get

$$2\{(L_U{}'F)(X, Y) + {}'F(L_U X, Y) + {}'F(X, L_U Y)\} =$$
$$= L_U(L_T G)(X, Y) + (L_T G)(L_U X, Y) + (L_T G)(X, L_U Y),$$

which due to (2.8) yields $2\lambda{}'F(X, Y) = L_T(\lambda G)(X, Y)$

i.e. $$(L_T G)(X, Y) = (T\lambda) G(X, Y) + \lambda(L_T G)(X, Y)$$

i.e. $$T\lambda = 0$$

which impies that λ = constant. Therefore it does not in general, admits a conformal infinitesimal automorphism, but when $\lambda = 0$, it admits a conformal infinitesimal automorphism.

Proposition 2.4. *If U is a conformal infinitesimal transformation of a compact almost paracontact metric manifold* $M^{2n+1}(n \geq 1)$ *and if* $G(T, [T, U])$ *has a fixed sign, then U is an infinitesimal automorphism.*

Proof. From $(2.5)_3$, we get $(L_U A)(T) = -\lambda A(T)$, which becomes $\lambda = -(L_U A)(T)$. Similarly from $(2.5)_4$, after a long calculation we get $\delta u = -(2n+1)\lambda$, where $u(X) = G(U, X)$ and $\delta u = \operatorname{div} U$. From these two equations, we get $\delta u = (2n+1)(L_U A)(T)$ which becomes

$$\delta u = (2n+1)\, G(T, [T, U]). \tag{3.1}$$

Since in a compact almost paracontact metric manifold the integral of the divergence of any vector field is zero, provided $G(T, [T, U])$ has a fixed sign, i.e. $\int_M \delta u = 0$, from (3.1) we have $\lambda = 0$, which proves the proposition.

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