PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 32 (46) 1982, pp 155—159

ALMOST CONTACT STRUCTURES INDUCED BY A CONFORMAL TRANSFORMATION

A. Sharfuddin, S. I. Husain

It has been shown that a conformal transformation on an odd-dimensional Riemannian manifold induces an Almost Contact Structure. This and other allied structures are studied in the present paper.

1. Introduction. Let M be a $(2m+1)$-dimensional almost contact metric manifold with structure tensors (φ, ξ, η, g). Then the structure tensors satisfy

$$
(1.1)\qquad \begin{aligned} &\varphi^2 X = -X + \eta(X)\xi, \ \eta(\xi) = 1, \ \eta(\varphi X) = 0, \ \varphi(\xi) = 0 \\ &g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \ \eta(X) = g(X, \xi), \end{aligned}
$$

for any vector fields X and Y on M. Further, if

$$
(1.2)\qquad 2F(X, Y) = (D_X\eta)(Y) - (D_Y\eta)(x)
$$

where $F(X, Y) = g\varphi(X, Y)$ and D is the Riemannian connexion of g, then M is called a contact metric manifold.

Let ∇ be a semi-symmetric metric connnexion in an almost contact metric manifold, that is, the torsion tensor T of ∇ is given by ([1], [3]) $T(X, Y) = \eta(Y)X - \eta(X)Y$, and further $(\nabla_X g)(Y, Z) = 0$. The Riemannian connexion D and the semi-symmetric metric connexion ∇ are related by [1]

$$
(1.3)\qquad \nabla_X Y = D_X Y + \eta(Y)X - g(X, Y)\xi.
$$

Let K and R denote the curvature tensors of D and ∇ respectively. Then it is seen that [2]

$$
(1.4)\quad R(X, Y, Z) = K(X, Y, Z) - \alpha(Y, Z)X + \alpha(X, Z)Y - g(Y, Z)\beta X + g(X, Z)\beta Y,
$$

where α is a tensor field of type (0, 2) defined by

$$
(1.5)\qquad \alpha(X, Y) = (\nabla_X\eta)(Y) - g(X, Y)/2
$$

and β is a tensor field of type (1, 1) defined by $g(\beta X, Y) = \alpha(X, Y)$.

In [2] we proved

Theorem (A). *The conformal curvature tensor of the semi-symmetric metric connexion on an almost contact metric manifold coincides with that of a Riemannian connexion.*

The above result leads us to say that an almost contact metric manifold with semi-symmetric metric connexion is conformally Riemannian, in the sense, that the semi-symmetric metric connexion gives rise to a connexion which is Riemannian with respest to a conformal metric. In the present paper, we have discussed the converse, viz., any odd-dimensional Riemannian manifold can be endowed with an almost contact metric structure induced by a conformal transformation. We have also defined here and studied a $(\hat{\varphi}, \hat{\xi}, \hat{\eta}, \hat{g})$-structure on the conformally transformed manifold.

2. Conformal connexion $\hat{\nabla}$. It is obvious that being non-symmetric the semi-symmetric metric connexion ∇ cannot be the Riemannian connexion of any metric conformal to g_{ij}. Nevertheless, Theorem (A) leads us to conscruct a symmetric connexion $\hat{\nabla}$ from ∇ which would be Riemannian with respect to a conformal metric. Such a connexion $\hat{\nabla}$ has been found to be $\hat{\nabla}_X Y = \nabla_X Y + \eta(X) Y$ or, equivalently

$$\hat{\nabla}_X Y = D_X Y + \eta(X) Y + \eta(Y) X - g(X, Y)\xi, \tag{2.1}$$

and it can be easily verified that $\hat{\nabla}$ is the Riemannian connexion with respect to the conformal metric $\hat{g}_{ij} = e^{2\eta_k x^k} g_{ij}$, where η is taken to be locally represented as $\eta = \eta_k dx^k$. In fact, we have

$$(\hat{\nabla}_X \hat{g})(Y, Z) = X \cdot \hat{g}(Y, Z) - \hat{g}(\hat{\nabla}_X Y, Z) - \hat{g}(Y, \hat{\nabla}_X Z),$$

which gives

$$(\hat{\nabla}_X \hat{g})(Y, Z) = X \cdot \{e^{2\eta_k x^k} g(Y, Z)\} - e^{2\eta_k x^k} \{g(\nabla_X Y + \eta(X) Y, Z) + g(Y, \nabla_X Z + \eta(X) Z)\} = 0.$$

Hence, in view of these observations, a study of this conformal connexion $\hat{\nabla}$ in almost contact metric manifolds appears worthwhile.

We begin with the following:

Theorem 2.1. *Let M be an almost contact metric manifold with the connexion $\hat{\nabla}$. Then*

$$\text{(a)}\ (\hat{\nabla}_X g)(Y, Z) = -2\eta(X) g(Y, Z), \quad \text{(b)}\ (\hat{\nabla}_{\varphi X} g)(Y, Z) = 0. \tag{2.2}$$

The proof can easily be obtained.

Remark. Equation (2.3) (a) shows a recurrence property of g with respect to the connexion $\hat{\nabla}$. In fact, $\hat{\nabla}$ becomes a Weyl-Hlavaty connexion [4] on (M, g).

We know that an almost contact metric manifold is a contact metric manifold if $F=d\eta$. But our η is so chosen that $d\eta=0$. Hence, we have

Theorem 2.2. *An almost contact metric manifold with the connexion $\hat{\nabla}$ can not be a contact metric manifold.*

Proof. Using the fact that the contact form η is closed, we get $F=0$ equivalent to $\varphi=0$ which proves the statement.

Let the curvature tensor corresponding to $\hat{\nabla}$ be

$$\hat{R}(X, Y, Z)=\hat{\nabla}_X\hat{\nabla}_Y Z-\hat{\nabla}_Y\hat{\nabla}_X Z-\hat{\nabla}_{[X, Y]} Z.$$

Then in almost contact metric manifold, we have

$$\hat{R}(X, Y, Z)=\hat{\nabla}_X(D_Y Z+\eta(Y)Z+\eta(Z)Y-g(Y, Z)\xi)$$
$$-\hat{\nabla}_Y(D_X Z+\eta(X)Z+\eta(Z)X-g(X, Z)\xi)$$
$$-D_{[X, Y]}Z-\eta(Z)[X, Y]-\eta([X, Y])Z+g([X, Y], Z)\xi,$$

from which, after some simplication, we get

(2.5) $\hat{R}(X, Y, Z)=K(X, Y, Z)-\alpha(Y, Z)X+\alpha(X, Z)Y-g(Y, Z)\beta X+g(X, Z)\beta Y,$

where

(2.6) $\alpha(Y, Z)=(\hat{\nabla}_Y\eta)(Z)-g(Y, Z)/2+\eta(Y)\eta(Z)=(\nabla_Y\eta)(Z)-g(Y, Z)/2$

and $g(\beta X, Y)=\alpha(X, Y)$, where $\beta Y=\hat{\nabla}_Y\xi-Y/2-\eta(Y)\xi=\nabla_Y\xi-Y/2$. Thus, we observe that the curvature tensors of $\hat{\nabla}$ and ∇ coincide, i.e., $\hat{R}(X, Y, Z)=$ $=R(X, Y, Z)$.

In consequence of η being a closed form, $\hat{R}(X, Y, Z)$ is seen to satisfy the following properties (cf [2], Theorem (2.2)).

(a) $\hat{R}(X, Y, Z)+\hat{R}(Y, Z, X)+\hat{R}(Z, X, Y)=0,$

(b) $(\hat{\nabla}_X\hat{R})(Y, Z, U)+(\hat{\nabla}_Y\hat{R})(Z, X, U)+(\hat{\nabla}_Z\hat{R})(X, Y, U)=0,$

(c) $\hat{\nabla}(X, Y, Z, U)+\hat{R}(X, Y, U, Z)=0,$

(d) $\hat{R}(X, Y, Z, U)-\hat{R}(Z, U, X, Y)=0.$

We also have (cf. [2], Theorem (4.2)).

Theorem 2.3. *If an almost contact metric manifold with the connexion* $\hat{\nabla}$ *is of constant sectional curvature, then it is conformally flat.*

3. Induced almost contact structures. In this section we consider the converse problem ef Theorem (A), viz., "Given any odd-dimensional Riemannian manifold, can we endow it with an almost contact metric structure induced by a conformal transformation?"

Let M be a $(2m+1)$-dimensional Riemannian manifold with metric tensor g and connexion D. Consider ρ to be a function of the local coordinates $\{x^k\}$ on M and effect a conformal transformation which gives a new metric tensor

$$\hat{g} = e^{2\rho} g. \tag{3.1}$$

Let us define a 1-form $d\rho$ and a vector field grad ρ (with respect to g):

$$\eta = d\rho, \text{ i.e., in local coordinates } \eta_k = \partial \rho / \partial x^k \tag{3.2}$$

$$\xi = \operatorname{grad} \rho, \text{ i.e., in local coordinates } \xi^i = g^{ij} \partial \rho / \partial x^j \tag{3.3}$$

Further, let us define an anti-symmetric (1,1)-tensor field φ on M locally satisfying the following condition:

$$\varphi^i_j \varphi^j_k + \delta^i_k = \frac{\partial \rho}{\partial x^k} \frac{\partial \rho}{\partial x^m} g^{im}. \tag{3.4}$$

Then we can easily prove

Theorem 3.1. *In a* $(2m+1)$*-dimensional Riemannian manifold* M*, the conformal transformation* $\hat{g} = e^{2\rho} g$ *induces an almost contact structure on* M *defined by* (φ, ξ, η, g) *as given by the equations* (3.2), (3.3) *and* (3.4).

This study thus establishes how an almost contact metric manifold with semi-symmetric metric connexion is conformally Riemannian and, conversely, how a conformal transformation on an odd-dimensional Riemannian manifold induces an almost contact structure. Let us put

$$F_{ki} = g_{im} \varphi^m_k, \text{ or equivalently, } F(X, Y) = g(\varphi X, Y), \tag{3.5.}$$

$$F_{ij} = -F_{ji}, \quad \text{or equivalently, } F(X, S) = -F(Y, X), \tag{3.6}$$

and

$$F_{ij} \varphi^i_k \varphi^j_m = F_{km}, \text{ or, } F(\varphi X, \varphi Y) = F(X, Y), \tag{3.7}$$

and

$$\varphi^i_j \varphi^j_k \varphi^k_1 + \varphi^i_1 = 0, \text{ or, } \varphi^3 + \varphi + 0$$

The following result is a direct consequence of the fact that the 1-form η is closed.

Theorem 3.2. *The almost contact metric manifold as defined in Theorem 3.1 can not be a contact metric manifold.*

4. $(\hat{\varphi}, \hat{\xi}, \hat{\eta}, \hat{g})$-structure. Let $(\hat{\varphi}, \hat{\xi}, \hat{\eta}, \hat{g})$ be an almost contact metric structure on a $(2m+1)$-dimensional manifold M induced by the conformal transformation $\hat{g}=e^{2\rho}g$, where $\rho=\eta_k x^k$ (defined previously). It is obvious that $(\varphi, \xi, \eta, \hat{g})$ does not define an almost contact metric structure on M. However starting with φ, ξ and η, we can construct $\hat{\varphi}$, $\hat{\xi}$ and $\hat{\eta}$ such that $(\hat{\varphi}, \hat{\xi}, \hat{\eta}, \hat{g})$ does give an almost contact metric structure to the corresponding transformed manifold $\hat{M}$ (say). This we do as follows: we set $\hat{\eta}=e^{\rho}\eta$, $\hat{\xi}=e^{-\rho}\xi$, $\hat{\varphi}=\varphi$. Then we have the following:

Theorem 4.1. *The collection $(\hat{\varphi}, \hat{\xi}, \hat{\eta}, \hat{g})$ gives an almost contact metric structure to the transformed manifold $\hat{M}$.*

Let us put $\hat{F}(X, Y)=\hat{g}(\hat{\varphi}X, Y)=g(\varphi X, Y)$. Then $\hat{F}(X, Y)+\hat{F}(Y, X)=0$, and $\hat{F}(\hat{\varphi}X, \hat{\varphi}Y)=\hat{F}(X, Y)$.

Theorem 4.2. *An almost contact metric manifold $\hat{M}$ can not be a contact metric manifold.*

The proof is direct.

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Department of Mathematics
Aligarh Muslim University
Aligarh — 202001, India.

(Received 21 05 1980)