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ON A WEAK COMMUTATIVITY CONDITION OF MAPPINGS IN FIXED POINT CONSIDERATIONS

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Abstract. A common fixed point theorem for a selfmapping of a complete metric space is proved: this theorem unifies and generalizes results of J. Danes [4] and K. M. Das, K. V. Naik [5].

1. Introduction. Let (X, d) be a complete metric space and $I: X \to X$ be the identity mapping of X. In [6], G. Jungck shows this interesting result:

Theorem 1. *Let f be a continuous selfmapping of (X, d). If there exists a mapping $g: X \to X$ and a constant $0 \leqslant \alpha < 1$ such that*

(A) $f(g(x)) = g(f(x))$ *for every* $x \in X$,

(B) $g(X) \subset f(X)$,

(C) $d(gx, gy) \leqslant \alpha\, d(fx, fy)$ *for every* $x, y \in X$,

then f and g have a unique common fixed point.

Note that if $f = I$, we obtain the well known Banach contraction principle. In recent years, several authors have generalized Theorem 1: for instance, Cheh-Chih Yeh [2], K. M. Das and K. V. Naik [5], M. S. Khan ([7] [8]) and Shih-Sen Chang [9]. In the present work, by retaining condition (B) and by replacing condition (C) with one due to J. Danes [4], we generalize Theorem 1 for the selmappings f and g of X satisfying a weaker hypothesis than commutativity (A), that is such that

$$(A_1) \qquad d(fgx, gfx) \leqslant d(fx, gx) \qquad \text{for every } x \in X.$$

Of course, our main theorem unifies and generalizes the results of [4] and [5].

2. Preliminaries. Let R^+ be the set of nonnegative real numbers and f, g be selfmappings of X such that (B) holds. Let now x_0 be an arbitrary point of X and $x_1 \in X$ such that $g(x_0) = f(x_1)$. Then, by induction, we can define a sequence $\{y_n\}_n^\infty = 0$ as follows

$$g(x_n) = f(x_{n+1}) = y_n \qquad n = 0, 1, 2, \ldots$$

By setting:

$$0(y_k, n) = \{y_k, y_{k+1}, \ldots y_{k+n}\} \qquad k = 0, 1, 2, \ldots$$

$$0(y_0, \infty) = \{y_0, y_1, \ldots y_n, \ldots\}$$

we denote with $\delta(0(y_k, n))$ and $\delta(0(y_0, \infty))$ the respective diameters. Furthermore, we put for every $x, y \in X$:

(1) $$M(x, y) = \max\{d(fx, fy), d(fx, gx), d(fy, gy), d(fx, gy), d(fy, gx)\}$$

and let $\varphi: R^+ \to R^+$ be a nondecreasing function, continuous from the right, such that $\varphi(t) < t$ for any $t > 0$.

In [4], J. Danes, by supposing that $f = I$ in (1), proves the following:

T h e o r e m 2. *Let $g: X \to X$ and $\varphi: R^+ \to R^+$ as above be such that for every $x, y \in X$:*

(C_1) $$d(gx, gy) \leqslant \varphi(M(x, y)).$$

If there exists a $x_0 \in X$ such that $\delta(0(x_0, \infty)) < \infty$, then g has a unique fixed point.

In [5], the authors. by generalizing the results of [3] and [6], give the following:

T h e o r e m 3 *Let f be a continuous selfmapping of X and $g: X \to X$ verifying conditions (A) and (B). If there exists a constant $0 \leqslant \alpha < 1$ such that for every $x, y \in X$:*

(C_2) $$d(gx, gy) \leqslant \alpha M(x, y)$$

then f and g have a unique common fixed point.

3. **Main result.** We first present some lemmas by modifying in some details the lemmas of [4] and [5].

L e m m a 1. *If $t_0 \in R^+$ and $t_k = \varphi(t_{k-1})$ for $k \geqslant 1$, then* $\lim t_k = 0$

P r o o f Obvious.

L e m m a 2. *Let f be a continuous selfmapping of (X, d) and g any selfmapping of X fulfilling conditions (B) and (C_1). Fok $k \geqslant 0$ and $n \geqslant 1$, let us suppose that $\delta(0(y_k, n)) > 0$ and $\delta(0(y_0, \infty)) < \infty$. Then, for $k \geqslant 1$, we have:*

$$\delta(0(y_k, n)) \leqslant \varphi^k(\delta(0(y_0, \infty))).$$

P r o o f. For i, j such that $k \leqslant i < j \leqslant k+n$, we have from ($C_1$):

$$d(y_i, y_j) = d(gx_i, gx_j) \leqslant \varphi(M(x_i, x_j)) \leqslant \varphi(\delta(0(y_{i-1}, j-i+1)))$$

where

$$M(x_i, x_j) = \max\{d(fx_i, fx_j), d(fx_i, gx_i), d(fx_j, gx_j), d(fx_i, gx_j), d(fx_j, gx_i)\} =$$
$$= \{d(y_{i-1}, y_{j-1}), d(y_{i-1}, y_i), d(y_{j-1}, y_j), d(y_{i-1}, y_j), d(y_{j-1}, y_i)\}.$$

Then

(2) $$\delta(0(y_k, n)) \leqslant \varphi(\delta(0(y_{i-1}, j-i+1))).$$

We claim $i = k$, otherwise if $i > k$, we have from (2) with $i-1 \geqslant k$ and $j \leqslant k+n$:

$$\delta(0(y_k, n)) \leqslant \varphi(\delta(0(y_{i-1}, j-i+1))) \leqslant \varphi(\delta(0(y_k, n))) < \delta(0(y_k, n));$$

a contradiction. By routine calculation, (2) implies

$$\delta(0(y_k, n)) \leqslant \varphi(\delta(0(y_{k-1}, j-k+1))) \leqslant \varphi(\delta(0(y_{k-1}, n+1))) \leqslant \varphi^k(\delta(0(y_0, n+k)))$$

and therefore the lemma follows because $\delta(0(y_0, \infty)) < \infty$ and φ is a nondecreasing function.

T h e o r e m 4. *Let f be a continuous selfmapping of (X, d) and g any selfmapping of X satisfying conditions (A_1) (B) (C_1).*

If there exists a $x_0 \in X$ such that $\delta(0(y_0, \infty) < \infty$, then f and g have a unique common fixed point.

P r o o f. We distinguish two cases:

i) if there exist $k \geqslant 0$ and $n \geqslant 1$ such that $\delta(0(y_k, n)) = 0$, we get immediately $y_k = y_{k+1}$, that is $f(x_{k+1}) = g(x_{k+1})$.

ii) if $\delta(0(y_k, n)) > 0$ for every $k \geqslant 0$ and $n \geqslant 1$, then by Lemma 1, given $\varepsilon > 0$, there is a $n_0 \geqslant 1$ such that $\varphi^{n_0}(\delta(0(y_0, \infty))) < \varepsilon$. Then, by Lemma 2, for $m > n \geqslant n_0$, we have $d(y_m, y_n) \leqslant \delta(0(y_{n_0}, m-n_0)) < \varepsilon$.

This means that $\{y_n\}_{n=0}^{\infty}$ is a Cauchy sequence and by completeness, there is a point z in X such that $z = \lim y_n$. Since f is continuous, $\{fy_n\}_{n=0}^{\infty}$ converges to fz.

Furthermore, (A_1) implies:

$$d(gy_n fz) \leqslant d(gy_n, fy_{n+1}) + d(fy_{n+1}, fz) = d(gfx_{n+1}, fgx_{n+1}) +$$

$$d(fy_{n+1}, fz) \leqslant d(fx_{n+1}, gx_{n+1}) + d(fy_{n+1}, fz) = d(y_n, y_{n+1}) + d(fy_{n+1}, fz).$$

Then, as $n \to \infty$, the above inequality guarantees that also $\{gy_n\}_{n=0}^{\infty}$ converges to fz. So

$$M(y_n, z) = \operatorname{Max}\{d(fy_n, fz), d(fy_n, gy_n), d(fz, gz), d(fy_n, gz), d(fz, gy_n)\}$$

converges to $d(fz, gz)$. By (C_1) we obtain $d(gy_n, gz) \leqslant \varphi(M(y_n, z))$ which, since φ is continuous from the right, implies $d(fz, gz) = \lim d(gy_n, gz) \leqslant \leqslant \lim\sup \varphi(M(y_n, z)) \leqslant \varphi(d(fz, gz))$, as $n \to \infty$ and therefore $d(fz, gz) = 0$ since $\varphi(t) < t$ *for every* $t > 0$.

In both cases, we have proved the existence of a point $z \in X$ such that

$$f(z) = g(z), \tag{3}$$

Then by (3) it follows that

$$fgz = gfz = ggz \tag{4}$$

Since (C_1) holds, (3) and (4) imply

$$d(ggz, gz) \leqslant \varphi(M(gz, z)) = \varphi(d(ggz, gz)).$$

Since $\varphi(t) < t$ for every $t > 0$, it must be that $d(ggz, gz) = 0$, that is gz is a fixed point of g. From (4), we deduce that gz is also a fixed point of f.

Let now w, z be two common distinct fixed points of f and g. Then

$$d(w, z) = d(gw, gz) \leqslant \varphi(M(w, z)) = \varphi(d(w, z)) < d(w, z);$$

a contradiction. This completes the proof.

Remarks. By setting $f = I$ in Theorem 4, we obtain Theorem 2. Moreover, if we put $\varphi(t) = \alpha t$ for every $t > 0$, Theorem 3 follows from Theorem 4 because Lemmas 2.1 and 2.2 of [5] imply that $\delta(0(y_0, \infty)) < \infty$ for every $x_0 \in X$.

In the following example Theorem 4 holds, whereas Theorem 3 (as well as the results of [2], [7], [8], [9]) is not applicable.

Example 1. Let $X = [0, 1]$ be with usual metric. Define $gx = x/2 + x$ and $fx = x/2$ for every $x \in X$. Further, let be $\varphi(t) = t/1 + t$ for every $t \geqslant 0$. We have:

$$g(X) = [0, 1/3], \; f(X) = [0, 1/2], \; \varphi(t) < t \text{ for } t > 0$$

and for every $x \in X$:

$$d(fgx, gfx) = \frac{x}{4+x} - \frac{x}{4+2x} = \frac{x}{(4+x)(4+2x)} \leqslant \frac{x^2}{4+2x} = \frac{x}{2} - \frac{x}{2+x} = d(fx, gx).$$

Moreover, for every $x, y \in X$:

$$d(gx, gy) = \left| \frac{x}{2+x} - \frac{y}{2+y} \right| = \frac{2|x-y|}{(2+x)(2+y)} \leqslant \frac{|x-y|}{2+|x-y|} = \varphi\left(\frac{|x-y|}{2}\right) =$$

$$= \varphi(d(fx, fy)) \leqslant \varphi(M(x, y)).$$

One can easily show that all the other assumptions of Theorem 4 are fulfilled.

Theorem 3 is not applicable because g does not commute with f being $gfx = x/(4+x) > x/(4+2x) = fgx$ for any $x \neq 0$ in X.

The example below shows that Theorem 4 is stronger than Theorem 3 for commuting maps.

Example 2. Let $X = (-\infty, \infty)$ be with usual metric. Define g, f: $X \to X$ as follows:

$$g(x) = \begin{cases} 0 & \text{if } x \leqslant 0 \\ x - x^2/2 & \text{if } 0 < x \leqslant 1 \\ 1/2 & \text{if } x > 1 \end{cases}$$

$$f(x) = \begin{cases} 0 & \text{if } x \leqslant 0 \\ x & \text{if } 0 < x \leqslant 1 \\ 1 & \text{if } x > 1 \end{cases}$$

$$\varphi(t) = \begin{cases} t - t^2/2 & \text{if } 0 \leqslant t \leqslant 1 \\ t/2 & \text{if } t > 1 \end{cases}$$

We have: $g(X) = [0, 1/2]$, $f(X) = [0, 1]$, $\varphi(t) < t$ for $t > 0$ and $fgx = gfx$ for every $x \in X$. Furthermore:

$$x \leqslant 0 \text{ and } y \leqslant 0 \text{ imply } d(gx, gy) = 0 = \varphi(d(fx, fy)),$$

$x \leqslant 0$ and $0 < y \leqslant 1$ imply $d(gx, gy) = y - y^2/2 = \varphi(d(fy, gx))$,

$x \leqslant 0$ and $y > 1$ imply $d(gx, gy) = 1/2 = \varphi(dfy, gx))$,

$0 < x \leqslant 1$ and $0 < y \leqslant 1$ imply $d(gx, gy) = |x-y| \cdot (1-(x+y)/2) \leqslant$

$$|x-y| \cdot (1 - |x-y|/2) = \varphi(d(fx, fy)),$$

$0 < x \leqslant 1$ and $y > 1$ imply $d(gx, gy) = 1/2 - x + x^2/2 \leqslant 1/2 - x^2/2 =$

$$(1-x) - (1-x)^2/2 = \varphi(d(fx, fy)),$$

$x > 1$ and $y > 1$ imply $d(gx, gy) = 0 = \varphi(d(fx, fy))$.

Summarizing, we have for every $x, y \in X$: $d(gx, gy) \leqslant \varphi(M(x, y))$, so (C_1) holds. All the other hypotheses of Theorem 4 are clearly satisfied, but this is not the case for Theorem 3. Indeed, suppose that (C_1) holds. We have for $x = 0$ and $0 < y \leqslant 1$:

$$y - y^2/2 \leqslant \alpha \operatorname{Max}\{y, 0, y^2/2, y - y^2/2, y\} = \alpha y,$$

that is $1 - y/2 \leqslant \alpha$ and, as $y \to 0$, consequently $1 \leqslant \alpha$; a contradiction.

The idea of this example appears in [1].

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