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SOME PERTURBATION RESULTS ON MULTIVALUED DIFFERENCE EQUATIONS

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Abstract If F and G are multivalued periodic upper semi-continuous maps and all solutions of $x(n+1)\in F(n, x(n))$ tend to zero, then there exists a sufficiently small $\varepsilon>0$ such that all solutions of $x(n+1)\in F(n, x(n))+\varepsilon G(n, x(n))$ are bounded. If moreover F and G are homogeneous, then the zero solution of the last equation is uniform stable.

1. Introduction and preliminaries. In this paper we study some perturbation results concerning multivalued homogeneous periodic difference equations. In our main result (Theorem 3) we show that, if all solutions of such an equation tend to zero asymptotically, then the zero solution of a suitably perturbed equation is uniform stable. For this purpose, previously, we prove some interesting results concerning boundedness and stability. Our method is based on that of Lasota and Strauss (1971), which is referred to multivalued autonomous homogeneous differential equations. Related results were given by De Blasi and Schinas, (1973, 1976), for the case of multivalued autonomous homogeneous difference equations and multivalued periodic homogeneous differential equations, respectively.

Denote by $N_{n_0}=\{n_0, n_0+1, \ldots\}$, n_0 any positive or zero integer, E a real Euclidean space with norm $|\cdot|$, $B(r)=B(0, r)$ ($\bar{B}(r)$) the open (closed) ball with center $0\in E$ and radius $r>0\,(r\geqq 0)$, $c(E)$ the set of nonempty compact subsets of E, $|A|=\sup\{|x|: x\in A\}$, $A\in c(E)$. In $c(E)$ addition and multiplication by nonnegative scalars are defined by $A+B=\{x+y: x\in A,\ y\in B\}$, $\lambda A=\{\lambda x: x\in A\}$.

A map $F: N_0\times E\to c(E)$, $(n, x)\to F(n, x)$, is called *upper semicontinuous* (u.s.c.)at $x\in E$, uniformly in n, if for every $\varepsilon>0$ there exists a $\delta=\delta(\varepsilon, x)>0$ such that $|x-y|<\delta$ implies $F(n, y)\subset F(n, x)+B(\varepsilon)$.

We denote by Φ_p the set of all maps $F: N_0\times E\to c(E)$, which are u.s.c. for all $x\in E$, uniformly in n, and p-periodic in $n\in N_0$ for all x, i.e., there exists a $p\in N_1$ such that $F(n+p, x)=F(n, x)$.

Let $n_0\in N_0$, $x_0\in E$, $F\in\Phi_p$. A function $x: N_{n_0}\to E$ is called *solution* of the multivalued difference equation

$$x(n+1)\in F(n, x(n)), \tag{1}$$

if $x(n_0)=x_0$ and $x(n)$ satisfies (1), for all $n \in N_{n_0+1}$. Such a solution will be denoted by $x(n, n_0, x_0)$.

Let $n_0 \in N_0$ and Q be a nonempty subset of E. For any $n \in N_{n_0}$, the set $R(n, n_0, Q)$ consisting of all points $y \in E$ such that there exists an $x_0 \in Q$ and a solution $x(n)$ of (1) with $x(n_0)=x_0$ and $x(n)=y$ is called *the set of attainability* of (1) starting from Q.

A map $F: N_0 \times E \to c(E)$, $F \in \Phi_p$, is called *(positively) homogeneous* with respect to x, if for any $s>0$, $F(n, sx)=sF(n, x)$, $(n, x) \in N_0 \times E$.

We denote by χ_p the set of all maps in Φ_p, which are homogeneous with respect to x.

In the sequel we need the following lemmas, which have been proved in De Blasi and Schinas (1973).

Lemma 1. *Let $F_1, F_2, \dots F_h \in \Phi_p$ (resp. χ_p). Then the maps defined by* $\sum_{i=1}^{h} F_i(n, x)$, $\bigcup_{i=1}^{h} F_i(n, x)$, *$(n, x) \in N_0 \times E$, are in Φ_p (resp. χ_p). Moreover if $F \in \Phi_p$ (resp. χ_p), also the map defined, for each $(n, x) \in N_0 \times E$, by* $\bigcup_{\varepsilon \in [0, \rho]} \varepsilon F(n, x)$, $0<\rho<1$, *is in Φ_p (resp. χ_p).*

Lemma 2. *Let $\{F_k\}$ be an infinite sequence of maps in Φ_p (resp. χ_p). such that $F_{k+1}(n, x) \subset F_k(n, x)$, $(n, x) \in N_0 \times E$, $k \in N_1$. The map defined, for each $(n, x) \in N_0 \times E$, by $F(n, x)=\bigcap_{k=1}^{\infty} F_k(n, x)$ is in Φ_p (resp. χ_p).*

Lemma 3. *For any $n_0 \in N_0$ and $Q \in c(E)$, the set of attainability of* (1) *$R(n, n_0, Q) \in c(E)$, for any $n \in N_{n_0}$.*

2. Boundedness of solutions. Theorem 1. *Let $Q_{p,r}=\{0, 1, \dots, p-1\} \times \bar{B}(r)$, $r>0$. Suppose that (i) $\{F_k\}$ is an infinite sequence of maps $F_k: N_0 \times E \to c(E)$, $F_k \in \Phi_p$,* and *$F_{k+1}(n, x) \subset F_k(n, x)$ $(n, x) \in N_0 \times E$, $k \in N_1$; (ii) every solution $x(n)=x(n, n_0, x_0)$, $(n_0, x_0) \in Q_{p,r}$, of* (1) *in which $F(n, x)=\bigcap_{k=1}^{\infty} F_k(n, x)$, approaches zero as $n \to \infty$, Then there exist $k \in N_1$ and $L \geq r$ such that every solution $x(n)=x(n, n_0, x_0)$, $(n_0, x_0) \in Q_{p,r}$,* of

$$(2_k) \qquad x(n+1) \in F_k(n, x(n))$$

satisfies $|x(n)| \leq L$, $n \in N_{n_0}$.

We omit the proof of the above theorem, since it is the discrete analogue of Theorem 1 of De Blasi and Schinas (1976). The only changes we need in our case are the following: (i) to consider the set $Q_{p,r}=\{0, 1, \dots, p-1\} \times \bar{B}(r)$ instead of Q of the above paper, (ii) to consider the number $|R_p|$

instead of H of the above paper, where R_p is defined by the relations $R_1(x_0)=F_k(0, x_0)$, $R_2(x_0)=\bigcup_{x\in R_1} F_k(1, x)$, ..., $R_p(x_0)=\bigcup_{x\in R_{p-1}} F_k(p-1, x)$, $R_p=\bigcup_{x_0\in \bar{B}(r)} R_p(x_0)$; and (iii) in order to prove that our corresponding $T_k\to\infty$, we apply Lemma 3.

Corollary 1. *Let F, $G\in\Phi_p$ and suppose that all solutions $x(n)=x(n, n_0, x_0)$ of* (1) *with $(n_0, x_0)\in Q_{p,r}$ tend to zero as $n\to\infty$. Then there exist $\varepsilon_1>0$ and $L\geqq r$ such that every solution $x(n)=x(n, n_0, x_0)$ of*

$$x(n+1)\in F(n, x(n))+\varepsilon G(n, x(n)), \quad 0\leqq\varepsilon\leqq\varepsilon_1, \tag{3}$$

with $(n_0, x_0)\in Q_{p,r}$ satisfies $|x(n)|\leqq L$, $n\in N_{n_0}$.

Proof. For any $k\in N_1$, define

$$F_k(n, x)=F(n, x)+\bigcup_{\varepsilon\in[0, 1/k]} \varepsilon G(n, x), \quad x\in E. \tag{4}$$

From Lemma 1, $F_k\in\Phi_p$. Since, moreover, $F_{k+1}(n, x)\subset F_k(n, x)$ and $\bigcap_{k=1}^{\infty} F_k(n, x)=F(n, x)$, hypotheses (i) and (ii) of Theorem 1 are fulfilled. From the same theorem there exist $k\in N_1$ and $L\geqq r$ such that every solution $x(n)=x(n, n_0, x_0)$ of (2_k) with $(n_0, x_0)\in Q_{p,r}$, where F_k is given by (4), satisfies $|x(n)|\leqq L$, $n\in N_{n_0}$. In particular, this inequality is satisfied by all solutions $x(n)=x(n, n_0, x_0)$ of (3) with $(n_0, x_0)\in Q_{p,r}$, where $0\leqq\varepsilon\leqq 1/k=\varepsilon_1$, because $F_k(n, x)\supset F(n, x)+\varepsilon G(n, x)$.

The next corollary is a particular application of Corollary 1 to the single-valued case.

Corollary 2. *Let f, $g: N_0\times E\to E$, f, $g\in\Phi_p$, and suppose that all solutions $x(n)=x(n, n_0, x_0)$ of*

$$x(n+1)=f(n, x(n)) \tag{5}$$

with $(n_0, x_0)\in Q_{p,r}$ tend to zero as $n\to\infty$. Then there exist $\varepsilon_1>0$ and $L\geqq r$ such that every solution $x(n)=x(n, n_0, x_0)$ of

$$x(n+1)=f(n, x(n))+\varepsilon g(n, x(n)), \quad 0\leqq\varepsilon\leqq\varepsilon_1, \tag{6}$$

with $(n_0, x_0)\in Q_{p,r}$ satisfies $|x(n)|\leqq L$, $n\in N_{n_0}$.

Example 1. Consider the multivalued difference equations

$$x(n+1)\in a(n+1)\bar{B}(|x(n)|), \tag{7}$$

$$x(n+1)\in a(n+1)\bar{B}(|x(n)|)+1/k\,\bar{B}(|x(n)|), \tag{8}$$

where $a(n)=1$, when n is even, $a(n)=\frac{1}{2}$, when n is odd. Clearly, all hypotheses of Corollary 1 are satisfied. Indeed, any solution $x(n)$ of (7) satisfies $|x(n)|\leqq 2^{-n/2}|x_0|$, when n is even, and $|x(n)|\leqq 2^{-(n+1)/2}|x_0|$, when n is odd, $n\in N_1$. So, there is $k\in N_1$ such that all solutions of (8) are bounded.

3. Uniform Stability. Theorem 2. *Let $Q_p=\{0, 1, \dots, p-1\}\times E$. Suppose that (i) $\{F_k\}$ is an infinite sequence of maps $F_k: N_0\times E\to c(E)$, $F_k\in\chi_p$ and $F_{k+1}(n, x)\subset F_k(n, x)$, $(n, x)\in N_0\times E$, $k\in N_1$;* (ii) every solution $x(n)=x(n, n_0, x_0)$, *$(n_0, x_0)\in Q_p$, of (1), in which $F(n, x)=\bigcap_{k=1}^{\infty} F_k(n, x)$, approaches zero as $n\to\infty$. Then there exist $k\in N_1$ and $L\geqq 1$ such that every solution $x(n)=x(n, n_0, x_0)$, $(n_0, x_0)\in Q_p$,* of *(2_k) satisfies,*

(9) $$|x(n)|\leqq L|x_0|, \quad n\in N_{n_0}.$$

P r o o f. By Theorem 1, there exist $k\in N_1$, $L\geqq 1$, such that every solution $z(n)$ of (2_k) with initial values in $Q_{p,1}$ satisfies

(10) $$|z(n)|\leqq L, \quad n\in N_{n_0}.$$

Let $x(n)=x(n, n_0, x_0)$ be any other solution of (2_k) with $(n_0, x_0)\in Q_p$ and $x_0\neq 0$. Set $z(n)=|x_0|^{-1}x(n)$. Dividing both sides of (2_k) by $|x_0|$ and taking into consideration the fact that F_k is homogeneous in x, we find that $z(n)$ is solution of (2_k). Since $(n_0, z_0)\in Q_{p,1}$, $z(n)$ satisfies (10) and so (9) holds for $x_0\neq 0$. When $x_0=0$, since F_k is homogeneous in x, it is easily seen that the only solution of (2_k) with initial values $(n_0, 0)$ is $x(n)\equiv 0$. This completes the proof.

T h e o r e m 3. *Let $F, G\in\chi_p$ and suppose that all solutions $x(n)=x(n, n_0, x_0)$ of (1) with $(n_0, x_0)\in Q_p$ tend to zero as $n\to\infty$. Then there exist $\varepsilon_1>0$ and $L\geqq 1$ such that every solution $x(n)=x(n, n_0, x_0)$ of (3) with $(n_0, x_0)\in Q_p$* satisfies $|x(n)|\leqq L|x_0|$, $n\in N_{n_0}$.

P r o o f. Define the maps F_k as in the proof of Corollary 1. From Lemma 1, $F_k\in\chi_p$. So, hypothesis (i) of Theorem 2 is satisfied for (4), and Theorem 2 applies. The conclusion of the proof is as in Corollary 1.

C o r o l l a r y 3. *Let $f, g: N_0\times E\to E$, $f, g\in\chi_p$, and suppose that all solutions $x(n)=x(n, n_0, x_0)$ of (5) with $(n_0, x_0)\in Q_p$ tend to zero as $n\to\infty$. Then there exists $\varepsilon_1>0$ and $L\geqq 1$ such that every solution $x(n)=x(n, n_0, x_0)$ of (6), with $(n_0, x_0)\in Q_p$, satisfies $|x(n)|\leqq L|x_0|$, $n\in N_{n_0}$.*

E x a m p l e 2. Consider the following equations in E

(11) $$x(n+1)\in a(n+1)\bar{B}(|x(n)|),$$

(12) $$x(n+1)\in a(n+1)\bar{B}(|x(n)|)+\beta(n)/k\cdot\bar{B}(|x(n)|),$$

where $a(n)=1$, $\beta(n)=0$, when n is even, and $a(n)=1/2$, $\beta(n)=1$, when n is odd. Then all hypotheses of Theorem 3 are satisfied, since, for all solutions $x(n)$ of (11), we have $|x(n)|\leqq 2^{-n/2}|x_0|$, when n is even, $|x(n)|\leqq 2^{-(n+1)/2}|x_0|$, when n is odd. For $k=2$, we find that all solutions of (12) satisfy the inequality $x(n)|\leqq|x_0|$. Note that, for $k\in N_3$, the zero solution of (12) is exponentially stable, because $|x(n)|\leqq((k+2)/2k)^{n/2}|x_0|$, when n is even, and $|x(n)|\leqq((k+2)/2k)^{(n+1)/2}|x|$, when n is odd.

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