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NOTES ON SOME GENERAL INEQUALITIES

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Abstract. In this paper we shall give some generalizations of the Steffensen, Jensen-Steffensen, Čebyšev and Hölder inequalities.

1. The generalization of the Steffensen inequality.

Theorem 1. *Let h be an integrable function positive on (a, b) and let f be an integrable function such that $x \mapsto f(x)/h(x)$ is a nondecreasing function on (a, b). If g is a real integrable function such that $0 \leqslant g(x) \leqslant 1$ $(\forall x \in (a, b))$, then*

$$\int_a^b f(x)\, g(x)\, dx \geqslant \int_a^{a+\lambda} f(x)\, dx \tag{1}$$

where λ is the solution of the equation

$$\int_a^{a+\lambda} h(x)\, dx = \int_a^b h(t)\, g(t)\, dt.$$

If $x \mapsto f(x)/h(x)$ is a nonincreasing function then the reverse inequality in (1) holds.

Proof. $\int_a^{a+\lambda} f(t)\,dt - \int_a^b f(t)\,g(t)\,dt =$

$$= \int_a^{a+\lambda} g(1-g(t))f(t)\,dt - \int_{a+\lambda}^b f(t)\,g(t)\,dt \leqslant$$

$$\leqslant \frac{f(a+\lambda)}{h(a+\lambda)} \int_a^{a+\lambda} h(t)(1-g(t))\,dt - \int_{a+\lambda}^b f(t)\,g(t)\,dt =$$

$$= \frac{f(a+\lambda)}{h(a+\lambda)} \left(\int_a^b h(t)\,g(t)\,dt - \int_a^{a+\lambda} h(t)\,g(t)\,dt\right) - \int_{a+\lambda}^b f(t)\,g(t)\,dt =$$

$$= \int_{a+\lambda}^b g(t)\,h(t)\left(\frac{f(a+\lambda)}{h(a+\lambda)} - \frac{f(t)}{h(t)}\right) dt \leqslant 0.$$

Thus the proof is finished.

By substitutions $g(x) \to 1-g(x)$, $\lambda \to b-a-\lambda$, Theorem 1. becomes:

Theorem 2. *Let the conditions of Theorem* 1. *be fulfilled. Then*

$$\int_a^b f(t)\,g(t)\,dt \leqslant \int_{b-\lambda}^b f(t)\,dt$$

where λ *is the solution of the equation*

$$\int_{b-\lambda}^b h(t)\,dt = \int_a^b h(t)\,g(t)\,dt$$

Remarks: 1° For $h(x)=1$, we have the well-known Steffensen inequalities (see, for example, [1, p. 105]).

2° Let f be n-convex function such that $f^{(k)}(a)=0$ $(k=0, 1, \dots, n-2)$ then $x \mapsto f(x)/(x-a)^{n-1}$ is nondecreasing function (see [2]), and using Theorem 1. we have that (1) is valid if

$$\lambda = \left(n \int_a^b (t-a)^{n-1} g(t)\,dt\right)^{1/n}$$

This result is given in [3]. The analogous result can be obtained for concave functions.

2. The generalization of the Jensen-Steffensen inequality.

Theorem 3. *Let* $f:[a, b] \to R$ *and* $H:[0, b-a] \to R$ *be differentiable functions such that* $x \mapsto f'(x)/H'(x-a)$ *is a nondecreasing function,* H *is an increasing function, and* $H(0)=0$. *If* a *is a monotonous n-tuple and* p *is a real n-tuple such that*

$$0 \leqslant P_k \leqslant P_n,\ P_n > 0 \quad \left(P_k = \sum_{i=1}^k p_i,\ k=1, \dots, n\right)$$

hold, then

$$f\left(a + H^{-1}\left(\frac{1}{P_n}\sum_{i=1}^n p_i H(a_i - a)\right)\right) \leqslant \frac{1}{P_n}\sum_{i=1}^n p_i f(a_i). \tag{2}$$

If $x \mapsto f'(x)/H'(x-a)$ *is a nonincreasing function then the reverse inequality in* (2) *holds.*

Proof. Let a be a nondecreasing n-tuple. By substitutions:

$f(t) \to f'(t)$, $h(t) \to H'(t-a)$ and $g(t)=g_i$ $(g_i = P_i/P_n)$ for $a_{i-1} < t \leqslant a_i$ $(a_0 = a)$,

from Theorem 1. we obtain Theorem 3.

Remarks: 3° If f and H are twice differentiable functions then the condition that $x \mapsto f'(x)/H'(x-a)$ is a nondecreasing function, can be replaced by the condition

$$f''(x)\,H'(x-a)-f'(x)\,H''(x-a)\geqslant 0.$$

This result, for $a=0$, is given in [4].

4° Let f be a $(k+1)$-convex function such that $f^{(m)}(a)=0$ $(m=1, \dots, k-1)$. Then $x \mapsto f'(x)/(x-a)^{k-1}$ is a nondecreasing function and (2) becomes (see [5] and [6]):

$$f\left(a+\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i(a_i-a)^k\right)^{1/k}\right)\leqslant\frac{1}{P_n}\sum_{i=1}^{n} p_i f(a_i).$$

If f is a $(k+1)$-concave function then the reverse inequality holds.

5° The condition for f in Theorem 3 can be weakened i.e. we can only suppose that function $x \mapsto f(a+H^{-1}(t))$ is convex (concave) on $[0, H(b-a)]$. This result, for $a=0$, is also given in [4].

3. The generalization of the Čebyšev inequality.

Theorem 4. *Let p be a nonnegative integrable function on $[a, b]$ and let f_j, h_j $(j=1, \dots, m)$ be integrable positive functions on $[a, b]$ monotonous in the same sense.*

(a) If $x \mapsto h_j(x)$ and $x \mapsto f_j(x)/h_j(x)$ $(j=1, \dots, m)$ are monotonous in the same sense, then

$$T(f_1, \dots, f_m; p)\geqslant T(h_1, \dots, h_m; p) \tag{3}$$

where

$$T(f_1, \dots, f_m; p)=\left(\int_a^b p(x)\,dx\right)^{m-1}\left(\int_a^b p(x)\prod_{j=1}^{m} f_j(x)\,dx\right)\left[\prod_{j=1}^{m}\left(\int_a^b p(x) f_j(x)\,dx\right)\right]^{-1}$$

(b) If $x \mapsto h_j(x)$ and $x \mapsto f_j(x)/h_j(x)$ $(j=1, \dots, m)$ are monotonous in the opposite sense, then the reverse inequality is valid.

Proof. (a) If $P(x)$ is a nonnegative integrable function and if $F(x)$ and $G(x)$ are monotonous functions in the same sense, then the well known Čebyšev inequality holds, i.e.

$$\int_a^b P(x)\,dx \int_a^b P(x)F(x)G(x)\,dx\geqslant \int_a^b P(x)F(x)\,dx \int_a^b P(x)G(x)\,dx. \tag{4}$$

If F and G are monotonous in the opposite sense then the reverse inequality is valid.

By substitutions: $P(x)=p(x)h_i(x)$, $F(x)=f_i(x)/h_i(x)$ and $G(x)=h_1(x)\cdots h_{i-1}(x)f_{i+1}(x)\cdots f_n(x)$ for $i=1, \ldots, m$ $(f_{m+1}(x)\equiv 1)$, from (4) we have

$$\int_a^b p(x)h_i(x)\,dx\int_b^b p(x)h_1(x)\cdots h_{i-1}(x)f_i(x)\cdots f_m(x)\,dx\geqslant \tag{5}$$
$$\geqslant\int_a^b p(x)f_i(x)\,dx\int_a^b p(x)h_1(x)\cdots h_i(x)f_{i+1}(x)\cdots f_m(x)\,dx.$$

Combining inequalities (5) for $i=1, \ldots, m$, we obtain (4).

(*b*) In this case the reverse inequality in (5) is valid, so the reverse inequality in (4) is also valid.

Analogously we can prove:

Theorem 5. *Let p be a nonnegative n-tuple and let a_j, b_j $(j=1, \ldots, m)$ be positive n-tuples monotonous in the same sense.*

(a) If b_j and a_j/b_j $(=(a_{j1}/b_{j1}, \ldots, a_{jn}/b_{jn}))$ are monotonous in the same sense, then

$$T(a_1, \ldots, a_m; p)\geqslant T(b_1, \ldots b_m; p)$$

where

$$T(a_1, \ldots, a_m; p)=\left(\sum_{i=1}^n p_i\right)^{m-1}\left(\sum_{i=1}^n p_i\prod_{j=1}^m a_{ji}\right)\left[\prod_{j=1}^m\left(\sum_{i=1}^n a_i a_{ji}\right)\right]^{-1}.$$

(b) If b_j and a_j/b_j are monotonous n-tuples in the opposite sense then the reverse inequality holds.

4. The generalization of the Hölder inequality.

Theorem 6. *Let q_j $(j=1, \ldots, m)$ be real numbers such that $0<q_j\leqslant 1$ $(1\leqslant j\leqslant m)$. If the conditions of Theorem 4 (a) are fulfilled, then*

$$H(f_1, \ldots, f_m; p)\geqslant H(h_1, \ldots, h_m; p) \tag{6}$$

where

$$H(f_1 \ldots f_m; p)=\int_a^b p(x)\prod_{j=1}^m f_j(x)\,dx\left[\prod_{j=1}^m\left(\int_a^b p(x)f_j(x)^{q_j}dx\right)^{1/q_j}\right]^{-1}.$$

If the conditions of Theorem 4 (b) are fulfilled then the reverse inequality in (6) holds.

Proof. This is a simple consequence of Theorem 5 from [7] and Theorem 4 (see [8]).

Using Theorems 1 and 2 from [7] and Theorem 5, we get:

Theorem 7. *Let q_j be defined as in the above theorem. If the conditions of Theorem 5 (a) are fulfilled then*

$$H(a_1, \ldots, a_m; p) \geqslant H(b_1, \ldots, b_m; p)$$

where

$$H(a_1, \ldots, a_m; p) = \sum_{i=1}^{n} p_i \left(\prod_{j=1}^{m} a_{ji}\right) \left[\prod_{j=1}^{m} \left(\sum_{i=1}^{n} p_i a_{ji}^{q_j}\right)^{1/q_j}\right]^{-1}$$

If the conditions of Theorem 5 (b) are fulfilled, then the reverse inequality is valid.

Remark. 6° In parts 3. and 4. we can only give two theorems with Lebesgue-Stieltjes' integrals. Proofs of these results and for analogous result from [7]are similar. For analogous generalization of Theorem 3 see Remark 5°.

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